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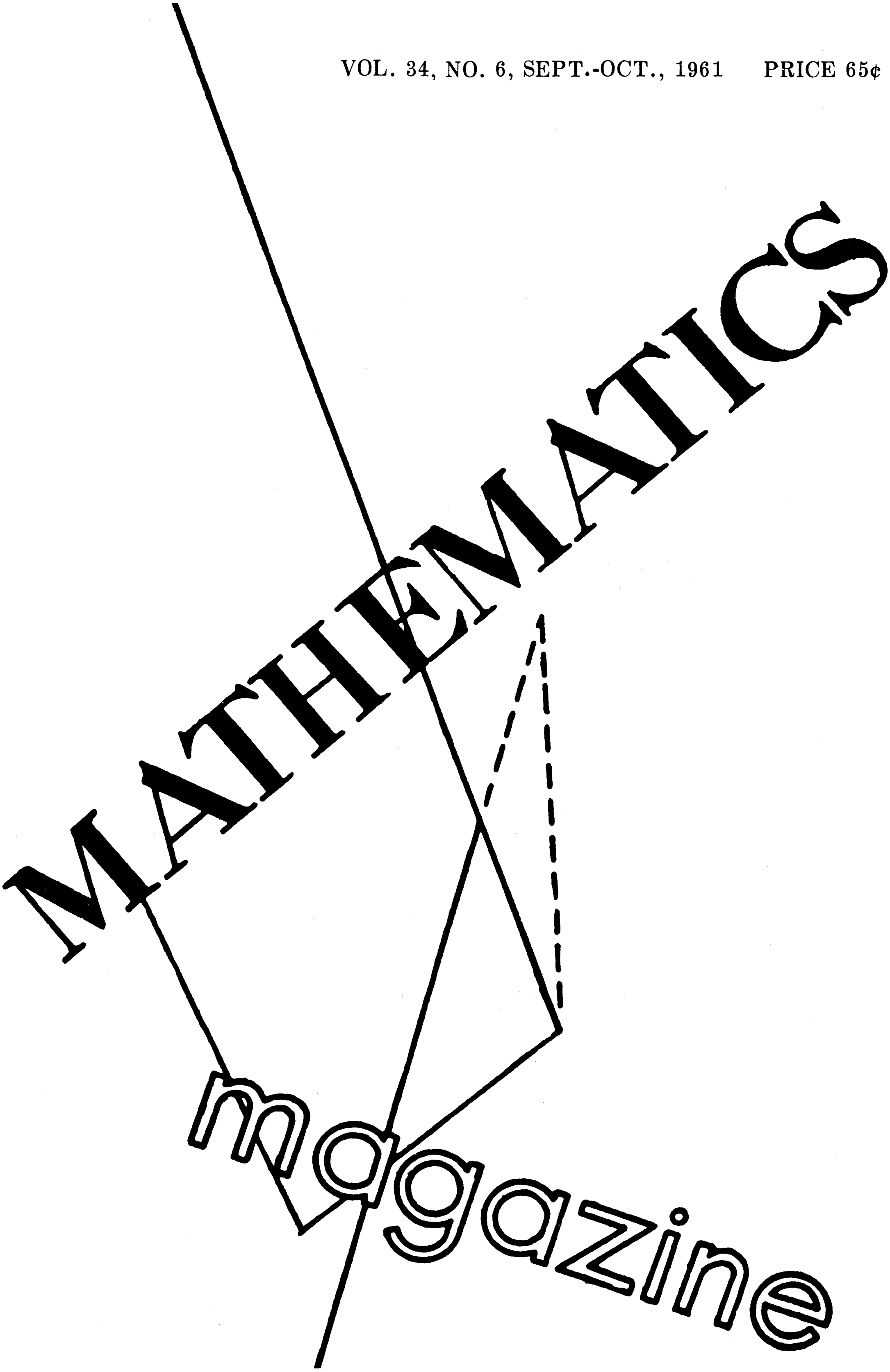

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MATHEMATICS MAGAZINE

ROBERT E. HORTON, Editor

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GLENN JAMES 1882-1961

On September 2, 1961, Glenn James, Editor of this Magazine from 1947 to 1960, died of heart failure.

A native of Indiana, he was a graduate of the University of Indiana (A.B. '05, A.M., '11) and of Columbia University (Ph.D. '17). He taught mathematics at Michigan State College from 1905-08, at Purdue University from 1908-19, and at the Carnegie Institute of Technology from 1919-22. In 1922 he joined the mathematics faculty at the University of California, Los Angeles, where he remained until his retirement in 1958.

One of James's unique contributions was the *Mathematics Dictionary*. The first edition, compiled with the collaboration of his son, Professor Robert C. James, was printed by the Digest Press. This venture was so successful, that a second edition involving a number of other collaborators, was published by Van Nostrand in 1949.

In 1945, Dr. James was perturbed by the fact that the *National Mathematics Magazine*, founded by S. T. Sanders, had ceased publication, after twenty years of service to the mathematical community. The problem of continuing the periodical was very difficult because of the lack of funds and of a sponsoring organization. However, with the help of several interested colleagues, particularly the late Prof. A. D. Michal of the California Institute of Technology, James succeeded in reviving the publication in 1947. The word "National" was dropped from the title in order to attract more foreign subscribers and authors. With doggedness, hard work, and ingenuity, James succeeded in keeping the *Magazine* going, in spite of many difficulties. When the price of commercial printing became too high for the income of the *Magazine*, he bought a vari-typer and trained several of his sons and daughters to use it. His wife, Inez, was the circulation manager (serving without pay). His faith and perseverance were rewarded in the following way. When ill health forced him to resign as Managing Editor, the Mathematical Association of America came to the rescue by agreeing to sponsor the *Magazine,* beginning in 1960.

Another interesting idea of Dr. James was the publication of a book containing an introduction to various fields of mathematics, with each chapter to be written by an expert in that field. The result was *The Tree of Mathematics*, published in 1957, by the Digest Press.

James had an original mind, and was a strong fighter for the principles he believed in. He was also a wonderful friend. He will be sorely missed by all who knew him, either personally, or through his voluminous correspondence with persons in various parts of the world.

D. H. Hyers

THE AMATEUR MATHEMATICIAN

Brother U. Alfred, F. S. C.

At a time when the professional mathematician has come into his own to a degree that would have been all but inconceivable twenty or twenty-five years ago, it may be opportune to offer a few thoughts about his counterpart, the amateur mathematician. Evidently, there is no point in trying to separate the entire human race into hard and fast sets of mathematician and non-mathematician and then of professional mathematician and amateur mathematician. There are all shades and degrees of proficiency and interest and it is quite conceivable that the professional may have an amateur spirit while the amateur may be engaged in professional work. Let us say then that we are not talking about specific human beings, but about the characteristics of professionalism and amateurism.

The professional mathematician is either engaged in using mathematics in his life work, whether as an actuary, a computation expert, a researcher in government or industry, or he is connected with some university or institute and seeking to extend the realm of knowledge. In either case, he is on the frontiers of some phase of mathematics and his greatest ambition in life is to make a contribution to mathematical knowledge or its application. For him, mathematics is a career. He keeps up with the latest research papers and spends a great deal of time trying to evolve or systematize new ideas. His moment of triumph comes with the publication of a paper which is a landmark in mathematical research or with the reading of a report before colleagues who have a clear perception of the depth and breadth of his ideas. His work appears in the most learned journals with a brief cataloging of similar efforts that preceded his contribution and then an abrupt attack on further developments, the whole bristling with an array of familiar or unfamiliar notations. But it is usually quite esoteric and cryptic. In fact, from the standpoint of the professional, it seems that the ideal is to compress the maximum of information into a minimum of space.

The amateur, on the other hand, is one who pursues mathematics because he enjoys it. For him, mathematics is not a career, but a way of life. Should he be fortunate enough in the course of his peregrinations to produce an original piece of work, he is understandably gratified, but he does not look upon this as the main purpose of his efforts. Consider, for example, the manner in which the professional and amateur go about their work. The professional will have to have at his disposal a well equipped and fairly complete library; the amateur, in many instances, is limited to a few fundamental works and a certain number of periodicals. If the professional gets an idea for a particular piece of research, his first thought is to examine the literature. What's the use of going through all the effort of rediscovering what others have already done? he reasons. The amateur, on the other hand, sets about investigating any problem that comes to mind should it happen to interest him. What difference does it make, he argues, whether this question was thought of and resolved by Gauss or Lagrange or Riemann years and years ago? This is not going to change the experience of discovery for me.

These last sentences state the thinking of the convinced amateur mathematician. Unfortunately, so much to-do has been made about who did what first and certain professionals take such a down-the-nose attitude toward the rediscovery of a mathematical idea, that the amateur is often ashamed to admit that he has come upon an old mathematical truth on his own. In fact, this is one of the major roadblocks to amateur activity. There is always that defeatist thought: somebody may have discovered this before.

And yet, in other fields of human activity, we seem to take it for granted that each human being has to rediscover everything for himself. There is no substitute for personally coming to appreciate Shakespeare, Beethoven, Grieg, Bizet, Raphael, or any of the other great creators of literature and art. In a similar fashion, each individual must rediscover mathematics for himself, either by coming to appreciate the ideas of others, or by arriving at these same ideas by his own efforts. Needless to say, the latter achievement is much more significant and thrilling. To discover an unusual idea which later turns out to be the original thought of an Archimedes, a Cayley, a Fermat, or some other mathematical great is like being able to sit down at the piano and produce a composition that bears a resemblance to the work of Beethoven or Tschaikovsky.

The amateur, then, welcomes any type of mathematical experience regardless of whether others have had it before him or not. Since he does not skip over the intermediate steps in order to get to the frontiers, it is quite possible that his appreciation of mathematical ramifications, nuances, and beauty may be keener than that of the professional. He may not know as much or be as far advanced, but what he does know he possesses with a certain fullness and richness.

One temptation the amateur has to meet and understand is undue concern about publication. If his thinking gradually shifts so that his first thought becomes: Will this effort lead to publication? then it is to be feared that he is drifting from love of mathematics to love of recognition. The psychological remedy for this deviation is the realization that genuine interest in mathematics will sooner or later lead to something that is worth sharing with others. Publication, in other words, is just a natural outgrowth of what the amateur does. He gets an idea that is new or exciting, either a completely original development or some light on old ideas. What is he going to do with it? Put it down on paper and lodge it in a folder in his files? The normal desire will be to share it with others. Unfortunately, this cannot be done in most instances by word of mouth. We do not have a society called SPEAK (Society for the Promotion and

Encouragement of Amateur Knowledge – in mathematics, of course) as we have an SPEBSQSA (Society for the Preservation and Encouragement of Barber Shop Quarter Singing in America) for fomenting more or less organized activity of the vocal cords. As a result, the amateur mathematician has to look about for some suitable magazine in which to have his work published.

If the amateur is able to sidestep the obstacles arising from professionalism, he may be bedeviled by other difficulties. One of the greatest deterrents is the thought: How do I know that this problem has a solution? Possibly, I can work on this for my entire life and still never arrive at an answer. Quite true. But if the history of mathematics tells us one thing more than another, it is this: Much of the progress in mathematics has come from the effort to solve what may be insoluble problems. Witness the hundreds and thousands of papers that have been written on Fermat's Last Theorem – and the search continues unabated to the present day. What does this mean for the amateur mathematician? It may well be that he may never solve the original problem he has attacked, but in the effort to do so, he will undoubtedly uncover many interesting ideas and developments that he would not have encountered otherwise. Furthermore, the more difficult the question, the greater the likelihood that it will lead to new and unfamiliar byways.

The mathematical amateur, in other words, has to be something of a Don Quixote attacking windmills. He does not go about looking for problems that fit neatly into his limited stock of mathematical tools. He is open to problems – period. Whether they can be solved or not, he tries to pry into them and thereby opens up new realms of thought.

NEED FOR MATHEMATICAL AMATEURS

The need for professional mathematicians is obvious. To speak of the need for mathematical amateurs may seem to be carrying a good idea to an absurd extreme. And yet, if mathematics is to be a living thing, it has to be something more than the esoteric sign language found in some of our mathematical publications. This is all very well for the favored few, the specialists in the various fields. But even they are not always on talking terms with each other, so that we are reminded of the situation that exists in certain islands of the Philippines where the people of one island cannot understand the language of the people of the island next door.

In addition, then, to specialists who are seeking to advance the frontiers of knowledge, there is a demand for a great many people who will be able to write and speak of mathematics in such a way as to make it appealing to the mass of those who have or might have an interest in the subject. Much excellent work has been done along these lines in recent years in the publication of interesting popular books on mathematics. But there is a need as well for suitable periodicals which will publish the work of amateur mathematicians and thus provide other amateurs with stimulating ideas at their level of comprehension.

The need for amateur mathematicians is particularly great in the teaching profession. Today, there is in the U. S. a tremendous ferment in mathematics teaching, especially at the secondary and primary levels. One of the major thoughts behind these new developments is that the teacher should get away from the old tell-them-and-test-them method in teaching mathematics and encourage the discovery or heuristic approach.

For this purpose, the teacher should realize that there is a certain type of textbook, well on the way to extinction let us hope, which does not represent living mathematics. What it contains is more like a collection of mathematical fossils arranged in orderly fashion as one might arrange the bones of a prehistoric dinosaur. The student can no more appreciate what constitutes living mathematics from these bones than he could form a realistic concept of a living animal from its skeleton.

When a teacher goes into class and announces: "Today, we are going to prove the theorem: If lines are drawn parallel to the base of a triangle, the sides are divided proportionally," he has offered an affront to the normal functioning of the human intellect. Understandably, students for centuries have been asking: "But where did this theorem come from?" The discovery approach helps the student see that mathematics is a living growth, a development of one idea from a previous idea. True, he may not cover as much ground, but at least he is learning some genuine mathematics and is developing a truly mathematical viewpoint. Furthermore, what he does know, he understands and in the long run, if he keeps consistently to this method, he will go much farther and deeper than he would have under the old system.

And now, if we may draw the obvious conclusion, the teacher who is to help the students discover their own mathematics should himself be a discoverer of mathematics. Without such a personal experience, he will find it very difficult to encourage and direct discovery in the minds of others. On the other hand, a teacher who has indulged in mathematicizing on his own will have something to offer besides what is in the textbook. There will be original viewpoints, new slants, interesting sidelights. His personal enthusiasm for mathematics, moreover, will be so manifest that it will carry over to his class.

But no doubt many of my readers, if they have arrived thus far in this article, will be thinking to themselves: Where does one get the time to do this? How does one find ideas to work on? Time....It seems to be a rather general rule that people always find time to do the things they really like. So, if a person has a genuine interest in mathematics, he will find time. Mathematics is one of those ideal fields which do not require a great deal of equipment, which can be pursued incidentally in going hither and yon, in hiking around the hills, or lying in bed. If mathematical ideas are on one's mind, they will grow and develop.

But where does one get the ideas? Personally, I am convinced that anybody with any familiarity with mathematics could sit down and write out a list of questions for further study. It seems that there are many things we do not understand about mathematics, but for the most part, we relegate these things to the back of the mind. One good method, then, for starting discovery work in mathematics is to put down any and all questions that intrigue or baffle us. After a while, one begins to consider the mathematical aspects of just about everything around him. And so, without too much delay, there will be no lack of topics to investigate.

Another very interesting approach may be mentioned. Everybody knows how, after a relatively short period of time following a course, what has been learned becomes rather hazy. With this shadowy remnant of knowledge and without the aid of a book, one can begin reconstructing ideas in the subject, browsing around in corners that were not covered during the formal course. It is amazing what a difference this can make in one's entire viewpoint. In other words, when we study from a text, the mind tends to be passive; we acquire ideas, but not too personally. When we set out on our own, we achieve in a way that makes these ideas a permanent possession of the mind.

CONCLUSION

In this article, a few thoughts have been offered regarding the amateur mathematician, the type of person for whom mathematics is an enjoyable pursuit, just as music, painting, or reading might be for others. It has been noted that if mathematics is to be something living, it must be found in the minds of people and not simply embalmed in textbooks or learned publications. Mathematical amateurs have a task of bringing the rich treasures of mathematical learning from the ivory tower to the school, the home and the market place. In particular, the teacher of mathematics, especially in the new era of mathematics teaching, will have to be something of an amateur if he is to be effective in developing genuine mathematicians.

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A THEOREM CONCERNING PRIME NUMBERS

Roger Crocker

It has been conjectured that every odd n from some point onward is the sum of a prime and a positive power of 2. The following theorem disproves this conjecture. The proof is more elementary and shorter than existing proofs of this theorem [1, 2].

Theorem. There is an infinity of odd numbers not the sum of a prime and a positive power of 2.

Proof: Take $2^{2^n}-5$, n an integer $\geqq 3$. Consider

$$(2^{2^n}-5)-2^a=(2^{2^n}-1)-4(1+2^{a-2}) , \quad a<2^n .$$

Now (for $a \geqq 3$) $1+2^{2^r} \mid 1+2^{a-2}$, where r is the largest power of 2 contained in $a-2$ ($r<n$; if a is odd, $r=0$). Now $2^{2^r}+1 \mid 2^{2^n}-1$ ($n \geqq 3$). Hence $2^{2^r}+1 \mid 2^{2^n}-5-2^a$, for $n \geqq 3$, $a \geqq 3$. Since

$$2^{2^n}-5-2^{2^n-1}=2^{2^n-1}-5>2^{2^{n-1}}+1 \quad (\text{for } n \geqq 3) ,$$

$$\frac{2^{2^n}-5-2^a}{2^{2^r}+1}>1 ,$$

and thus $2^{2^n}-5-2^a$ is composite ($n \geqq 3$) for $a \geqq 3$. If $a=1$, $3 \mid 2^{2^n}-5-2$ as $3 \mid 2^{2^n}-1-6$ for $n \geqq 3$. If $a=2$,

$$2^{2^n}-5-2^2=2^{2^n}-9=(2^{2^{n-1}}+3)(2^{2^{n-1}}-3) ,$$

both factors >1 for $n \geqq 3$.

Q.E.D.

Conjecture. Every positive integer $n \geqq 3$ is the sum of a prime and a fixed number s of non-negative k^{th} powers (for a given $k \geqq 2$), where s depends only on k.

This theorem follows immediately from Waring's problem and Hilbert's Theorem. Now let $P(k)$ be the least value of s for which it is true that every *sufficiently large* number can be represented as the sum of a prime and s k^{th} powers; let $G(k)$ be the corresponding value of s in the

(Continued on page 344.)

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Waring problem. Then it is conjectured:

$$P(k) \leqq \left[\frac{G(k)}{2}\right]$$

for $k \neq 2^a$.

$$P(k) \leqq G(k) - k$$

for $k = 2^a$.

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SOME RELATIONS INVOLVING THE FINITE HARMONIC SERIES

H. W. Gould

1. *Introduction.* R. R. Goldberg [1] proposed the following finite summation recently

$$(1)\qquad n^2\sum_{k=0}^{n}(-1)^{n+k}\frac{(n+k-1)!}{(n-k)!\,k!\,k!}\sum_{j=1}^{n+k-1}\frac{1}{j}=1,$$

and a published solution of Chih-yi Wang [2] made use, essentially, of the novel device that $n^2 = (n+k)(n-k)+k^2$ in order to effect a proof of (1). This surely allows a short proof, however it may be of interest to see what may motivate writing down (1) to begin with. To an expert in work with series it is obvious why we might expect some such relation as (1) to hold, namely that by differentiating some more general formula we get (1) as a special case. To one who is accustomed to making guesses this way it is also at once clear that (1) and the following rather well-known formula

$$(2)\qquad \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\sum_{j=1}^{k}\frac{1}{j}=\frac{1}{n}$$

must be related. We should like therefore to give the more general formulas which come to mind and show how one goes about the manipulations involved, as it requires apparently involved work to show what we propose here.

2. *Basic formulas.* We shall need to observe that

$$(3)\qquad \binom{x+n}{n}=\prod_{k=1}^{n}\frac{x+k}{k},$$

this being true for all real values of x, and integers $n \geqq 0$ if we take the product to be unity when $n = 0$.

From this we find by the usual rule for differentiating a product

$$(4)\qquad D_x\binom{x+n}{n}=\binom{x+n}{n}\sum_{j=1}^{n}\frac{1}{j+x},$$

which we shall see is the clue to all that follows.

First we shall need a fairly general binomial coefficient identity which we shall prove simply from the Vandermonde convolution.

Since

$$\binom{y}{n}\binom{n}{j} = \binom{y}{j}\binom{y-j}{n-j}$$

we see that

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{z}{j}\binom{y}{n}\binom{y}{j}^{-1} = \sum_{j=0}^{n} (-1)^j \binom{z}{j}\binom{y-j}{n-j}$$

$$= (-1)^n \sum_{j=0}^{n} \binom{z}{j}\binom{-y+n-1}{n-j}$$

$$= (-1)^n \binom{z-y+n-1}{n} = \binom{y-z}{n}.$$

Here we made use of the Vandermonde convolution formula and the identity

$$\binom{x}{n} = (-1)^n \binom{-x+n-1}{n}.$$

If we next replace y in this by $-x-1$ we find that

$$(5) \qquad \sum_{j=0}^{n} \binom{n}{j}\binom{z}{j}\binom{x+j}{j}^{-1} = \binom{x+z+n}{n}\binom{x+n}{n}^{-1}.$$

In this set $z = -1$ for our particular application. The result we get is that

$$(6) \qquad \sum_{j=0}^{n} (-1)^k \binom{n}{k}\binom{x+k}{k}^{-1} = \frac{x}{x+n}.$$

If we differentiate each member here, using (4), and set $x = 0$ we find that we have proved relation (2).

3. *Goldberg's series.* It seems difficult to put in reasonable language why the following sequence of steps is tried, but we shall attempt to explain it.

First rewrite Goldberg's formula in binomial notation:

$$(7) \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+k-1}{k} \sum_{j=1}^{n+k-1} \frac{1}{j} = \frac{(-1)^n}{n}.$$

It appears that this must come from (5) but the upper limit of summation for j, namely $n+k-1$ has to be replaced by k, which is in fact possible. To be specific we shall prove that if the sum in (7) be called S, then

$$(8) \qquad S = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+k-1}{k} \sum_{j=1}^{k+1} \frac{1}{j}, \quad \text{for } n \geqq 3.$$

We note first that

$$\sum_{j=1}^{n+k-1} \frac{1}{j} = \sum_{j=1}^{k+1} \frac{1}{j} + \sum_{j=0}^{n-3} \frac{1}{j+k+2} \,.$$

With this we split S as given to us, into two parts, the desired part and another part which we show is identically zero.

In order to accomplish this without too much effort we need the easily proved lemma that if $f(x)$ be a polynomial in x of degree $\leqq n$, then [3]

$$f(x+y) = y\binom{y+n}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{f(x-k)}{y+k} \,. \tag{9}$$

In this choose $f(x)$ to be

$$\binom{2n-x-1}{n-1} \,.$$

Then we have

$$f(x-k)\Big|_{x=n} = \binom{n+k-1}{n-1} = \binom{n+k-1}{k} \,,$$

Using this in (9) we obtain then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+k-1}{k} \frac{1}{y+k} = \binom{n-y-1}{n-1} y^{-1} \binom{y+n}{n}^{-1} \,. \tag{10}$$

In this if we let $y = j+2$, and observe that the numerator on the right

$$\binom{n-j-3}{n-1} = 0 \quad \text{for } 0 \leqq j \leqq n-3 \,, \quad n \geqq 3 \,,$$

then it follows that if we sum the expression

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+k-1}{k} \frac{1}{j+k+2}$$

from $j = 0$ to $j = n-3$ for $n \geqq 3$, the result must be zero, and so the second part of the split sum is zero, provided $n \geqq 3$.

We next must show that (8) then may be gotten from (5). To accomplish this, we apply the differentiation technique again, and see that (5) yields most generally

$$\sum_{k=1}^{n} \binom{n}{k}\binom{z}{k}\binom{x+k}{k}^{-1} \sum_{j=1}^{k} \frac{1}{j+x} = \binom{x+z+n}{n}\binom{x+n}{n}^{-1} \left\{ \sum_{j=1}^{n} \frac{1}{j+x} - \sum_{j=1}^{n} \frac{1}{j+x+z} \right\} . \tag{11}$$

In this we let $x = 0$ and now instead of letting $z = -1$ as we did when we proved (2), we let z tend to $-n$. It is easily verified that this calculation yields the relation

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{n+k-1}{k}\sum_{j=1}^{k}\frac{1}{j}=\frac{(-1)^n}{n}, \tag{12}$$

which is curiously close to (8), which we shall now show has the same value.

Indeed, for the sum S in (8) we have

$$S=1+\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{n+k-1}{k}\sum_{j=1}^{k+1}\frac{1}{j}$$

$$=1+\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{n+k-1}{k}\sum_{j=1}^{k}\frac{1}{j}+\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{n+k-1}{k}\frac{1}{k+1}$$

$$=1+\frac{(-1)^n}{n}+\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k-1}{k}\frac{1}{k+1}-1,$$

by (12)

$$=\frac{(-1)^n}{n},$$

since the other summation is identically zero as may be seen again as a special case of (10). Therefore the desired result is obtained.

The writer is of the opinion that to alter or simplify these steps would camouflage the basic technique with which one may operate successfully in a problem such as this. The technique is highly involved and can be valuable in analyzing random problems one encounters.

Naturally, interesting formulas of other kinds may be found from relation (11), and though we shall not go into it, we observe that the relations

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\sum_{j=1}^{2k}\frac{1}{j}=\frac{1}{2n}+\frac{2\cdot 4\cdots(2n-2)}{3\cdot 5\cdots(2n-1)},$$

$$\sum_{k=1}^{2n-1}(-1)^{k-1}\binom{2n}{k}^{-1}\sum_{j=1}^{k}\frac{1}{j}=\frac{n}{2(n+1)^2}+\frac{1}{2n+2}\sum_{k=1}^{2n}\frac{1}{k},$$

and

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\binom{2k}{k}^{-1}\frac{2^{2k}}{k}=2\sum_{j=1}^{2n}\frac{1}{j}-\sum_{j=1}^{n}\frac{1}{j},$$

the last of which may be gotten from the interesting formula

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{x+k}{k}\binom{2k}{k}^{-1}\frac{x2^{2k}}{x+k} = (-1)^n\binom{2x}{2n}\binom{x}{n}^{-1},$$

follow in a similar fashion through the application of the differentiation technique. The three last mentioned formulas were proposed as problems in the *Nordisk Matematisk Tidskrift*, Volumes 29 and 30(1948). Our proofs are not short and we omit them.

The writer understands from conversation with Karl Goldberg (National Bureau of Standards) that he has a very short elegant proof of formula (1) using properties of Legendre polynomials, obtaining first more general results.

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CURIOSA FROM 1961

Charles W. Trigg

1) $1961 = (1 \cdot 9 \cdot 6 - 1)(-19 + 61 - 1 - 9 + 6 - 1)$.
2) $1961 = -12 + (34)(56) + 78 - 9$.
3) $1961 = 987 + 654 + 321 - 0!$
4) $1961 = 0! + 1! + 2(3!)(4!) + 5 \cdot 6 \cdot 7 \cdot 8 - 9$.
5) $1961 = 6^4 + 5^4 + (3^4 - 1^4)/2$.
6) $1961 = 3^7 - 15^2 - 1^2$.
7) $1961 = 5^5 - 2^{10} - 2^7 - 2^3 - 2^2$.
8) $1961 \doteq 1960.78 \doteq 10^5/(-1-9+61)$.
9) $(1!9!6!1!)(!1!9!6!1) = 0$, where $!x$ is sub-factorial x.
10) $(1-1)96 = 19 + 96 + 61 - 16 - 69 - 91 = 1^9 - 1^6 = 0$.

Los Angeles City College

CRYPTOGRAPHIC SLIDE RULES

Jack Levine

1. *Introduction.* The use of mechanical and other devices to facilitate cryptographic and cryptanalytic procedures is well known. In particular, sliding strips inscribed with alphabetic or numerical sequences form an almost indispensable feature of the cryptographic art.

We describe here two simple types of sliding strips arranged in slide rule form to be used as aids in the algebraic encipherment (and decipherment) process of Hill [1, 2], (see also Levine [3]). In addition, an alignment chart to achieve the same purpose is described. For simplicity we restrict ourselves to the case of $n = 2$ congruences, corresponding to the digraphic encipherment.

For the benefit of those unfamiliar with the stated cryptographic method we give below a brief description of the case $n = 2$.

Some permutation of the alphabet letters is selected, and its letters numbered in order 0, 1, 2, ⋯, 25. We select the permutation:

(1.1)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
	D	*R*	*B*	*U*	*E*	*W*	*G*	*F*	*X*	*H*	*Z*	*K*	*O*	*N*	*M*	*L*	*P*	*I*	*Q*	*A*	*T*	*C*	*V*	*Y*	*J*	*S*

The inverse correspondence is also used, this being given by

(1.2)	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*	*M*	*N*	*O*	*P*	*Q*	*R*	*S*	*T*	*U*	*V*	*W*	*X*	*Y*	*Z*
	19	2	21	0	4	7	6	9	17	24	11	15	14	13	12	16	18	1	25	20	3	22	5	8	23	10

Two congruences are selected of the form

$$C_1 \equiv aP_1 + bP_2\ , \quad C_2 \equiv cP_1 + dP_2 \quad (\text{mod } 26)\ , \tag{1.3}$$

with determinant $ad - bc$ prime to 26. In this article we shall use the case

$$C_1 \equiv 9P_1 + 8P_2\ , \quad C_2 \equiv 16P_1 + 17P_2 \quad (\text{mod } 26)\ . \tag{1.4}$$

Now, to encipher a text, as SLIDE RULE, divide the letters in pairs, and replace each letter by its numerical value according to (1.2):

(1.5)	*S*	*L*	*I*	*D*	*E*	*R*	*U*	*L*	*E*	*X*
	25	15	17	0	4	1	3	15	4	8

In (1.4) replace P_1P_2 by the numerical pairs 25 15, 17 0, ⋯, and calculate C_1C_2. Then replace the numbers C_1C_2 by their letter values using (1.1). We have for pair $E\ R = 4\ 1$,

$$C_1 \equiv 9(4) + 8(1) = 44 \equiv 18 = Q\ ,$$

$$C_2 \equiv 16(4) + 17(1) = 81 \equiv 3 = U\ .$$

Hence $E\ R$ is enciphered to $Q\ U$. The complete encipherment of (1.5) is found to be *FW YO QU II BC*, or *FWYOQ UIIBC*, as is customarily written.

To decipher, we use the inverse of (1.3) and proceed as for encipherment. For purposes of this paper we have chosen (1.4) to be involutory, i. e.,

$$(1.6) \qquad P_1 \equiv 9C_1+8C_2\,, \quad P_2 \equiv 16C_1+17C_2 \pmod{26}\,,$$

so that the same slide rule or chart can be used both for encipherment and decipherment.

2. *First form of slide rule*. We first describe the rule to be used with (1.4). The notation P_1 (or P_2) will indicate both a letter and its numerical value according to (1.1) or (1.2).

Place

$$(2.1) \quad u_1 \equiv 9P_1\,, \quad u_2 \equiv 8P_2\,, \quad v_1 \equiv 16P_1\,, \quad v_2 \equiv 17P_2 \pmod{26}\,,$$

so that $C_1 \equiv u_1+u_2$, $C_2 \equiv v_1+v_2$.

Let P_1, P_2 take the values 0, 1, 2, ⋯, 25, and calculate the u's and v's by (2.1).

P_i	*D*	*R*	*B*	*U*	*E*	*W*	*G*	*F*	*X*	*H*	*Z*	*K*	*O*	*N*	*M*	*L*	*P*	*I*	*Q*	*A*	*T*	*C*	*V*	*Y*	*J*	*S*
P_i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
u_1	0	9	18	1	10	19	2	11	20	3	12	21	4	13	22	5	14	23	6	15	24	7	16	25	8	17
u_2	0	8	16	24	6	14	22	4	12	20	2	10	18	0	8	16	24	6	14	22	4	12	20	2	10	18
v_1	0	16	6	22	12	2	18	8	24	14	4	20	10	0	16	6	22	12	2	18	8	24	14	4	20	10
v_2	0	17	8	25	16	7	24	15	6	23	14	5	22	13	4	21	12	3	20	11	2	19	10	1	18	9

Fig. 1.

The 26 columns of the above table are now rearranged according to the normal alphabet sequence $ABC \cdots XYZ$:

P_i	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*	*M*	*N*	*O*	*P*	*Q*	*R*	*S*	*T*	*U*	*V*	*W*	*X*	*Y*	*Z*
u_1	15	18	7	0	10	11	2	3	23	8	21	5	22	13	4	14	6	9	17	24	1	16	19	20	25	12
u_2	22	16	12	0	6	4	22	20	6	10	10	16	8	0	18	24	14	8	18	4	24	20	14	12	2	2
v_1	18	6	24	0	12	8	18	14	12	20	20	6	16	0	10	22	2	16	10	8	22	14	2	24	4	4
v_2	11	8	19	0	16	15	24	23	3	18	5	21	4	13	22	12	20	17	9	2	25	10	7	6	1	14

Fig. 2.

A slide rule is now constructed with fixed upper and lower parts, and a middle sliding part: the upper part contains in order the P_1-scale (normal alphabet), and the u_1-scale; the sliding middle part contains in order the u_2-scale, the P_2-scale, and the v_2-scale; the fixed bottom part contains the v_1-scale. The three parts with scales are shown in Fig. 3. To simplify computations, numbers $x > 13$ are replaced by $x-26$, so 18 becomes -8, indicated by $\overline{8}$.

P_1	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
u_1	$\overline{11}$	$\overline{8}$	7	0	10	11	2	3	$\overline{3}$	8	$\overline{5}$	5	$\overline{4}$	13	4	$\overline{12}$	6	9	$\overline{9}$	$\overline{2}$	1	$\overline{10}$	$\overline{7}$	$\overline{6}$	$\overline{1}$	12
u_2	$\overline{4}$	$\overline{10}$	12	0	6	4	$\overline{4}$	$\overline{6}$	6	10	10	$\overline{10}$	8	0	$\overline{8}$	$\overline{2}$	$\overline{12}$	8	$\overline{8}$	4	$\overline{2}$	$\overline{6}$	$\overline{12}$	12	2	2
P_2	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
v_2	11	8	$\overline{7}$	0	$\overline{10}$	$\overline{11}$	$\overline{2}$	$\overline{3}$	3	$\overline{8}$	5	$\overline{5}$	4	12	$\overline{4}$	12	$\overline{6}$	$\overline{9}$	9	2	$\overline{1}$	10	7	6	1	$\overline{12}$
v_1	$\overline{8}$	6	$\overline{2}$	0	12	8	$\overline{8}$	$\overline{12}$	12	$\overline{6}$	$\overline{6}$	6	$\overline{10}$	0	10	$\overline{4}$	2	$\overline{10}$	10	8	$\overline{4}$	$\overline{12}$	2	$\overline{2}$	4	4

Fig. 3.

To encipher $P_1P_2 = SL$, set $P_2 = L$ of the P_2-scale opposite $P_1 = S$ of the P_1-scale:

P_1	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	···
u_1	$\overline{11}$	$\overline{8}$	7	0	10	11	2	3	$\overline{3}$	8	$\overline{5}$	5	$\overline{4}$	13	4	$\overline{12}$	6	9	$\overline{9}$	$\overline{2}$	1	···
u_2								$\overline{4}$	$\overline{10}$	12	0	6	4	$\overline{4}$	$\overline{6}$	6	10	10	$\overline{10}$	8	0	···
P_2								A	B	C	D	E	F	G	H	I	J	K	L	M	N	···
v_2								11	8	$\overline{7}$	0	$\overline{10}$	$\overline{11}$	$\overline{2}$	$\overline{3}$	3	$\overline{8}$	5	$\overline{5}$	4	12	···
v_1	$\overline{8}$	6	$\overline{2}$	0	12	8	$\overline{8}$	$\overline{12}$	12	$\overline{6}$	$\overline{6}$	6	$\overline{10}$	0	10	$\overline{4}$	2	$\overline{10}$	10	8	4	···

Add the two numbers between the S and L on the u_1, u_2 scales respectively. These give $\overline{9}+\overline{10} = \overline{19} = 7 = F = C_1$. Add the two numbers under the L on the v_1, v_2 scales respectively. These give $\overline{5}+10 = 5 = W = C_2$. Hence $C_1C_2 = FW$, is the encipherment of SL.

The encipherment (or decipherment) of any pair proceeds in a similar manner.

The general setting would appear as

A B C · · · P_1 ···	
· · · · · · · u_1 ···	
	· · · · ··· u_2 ···
	A B C ··· P_2 ···
	· · · · ··· v_2 ···
· · · · · · · v_1 ···	

The construction of the rule for the general case (1.3) is evident. Corresponding to Fig. 1 we would have

	A_1	A_2	A_3	·	·	···	A_{24}	A_{25}
P_1	0	1	2	3	4	···	24	25
u_1	0	a	$2a$	$3a$	$4a$	···	$24a$	$25a$
u_2	0	b	$2b$	$3b$	$4b$	···	$24b$	$25b$
v_1	0	c	$2c$	$3c$	$4c$	···	$24c$	$25c$
v_2	0	d	$2d$	$3d$	$4d$	···	$24d$	$25d$

where $A_1A_2 \cdots A_{24}A_{25}$ is the selected permutation (corresponding to (1.1)). The columns are then rearranged as in Fig. 2, and the rule constructed as described above.

3. *Second form of slide rule.* Again, we first describe the construction with reference to (1.4). As in the first rule there is a fixed upper and lower part, and a sliding middle part. There are five scales each consisting of the sequence

$$0 \ 1 \ 2 \cdots 24 \ 25 \ 0 \ 1 \ 2 \cdots 25 \tag{3.1}$$

Each of these five sequences is to be replaced by letters according to the following scheme.

Upper fixed part:

scale (a), sequence (3.1) replaced by letter sequence (1.1),

scale (b), sequence (3.1) represents the u_1 values in $u_1 \equiv 9P_1$. These u_1 values are to be replaced by the corresponding P_1 (letter) values. Thus, $u_1 = 0$, $P_1 = 0 = D$; $u_1 = 1$, $P_1 = 3 = U$; $u_1 = 2$, $P_1 = 6 = G$, etc. Hence $D \ U \ G \cdots$ forms scale (b).

Middle sliding part:

scale (c), (3.1) represents the u_2 values in $u_2 \equiv 8P_2$. These u_2 values are to be replaced by the corresponding P_2 values. Each even u_2 gives two P_2 values, each odd u_2 gives none:

$$u_2 = 0\,, \quad P_2 = 0 \quad \text{or} \quad 13 = D \ N$$
$$u_2 = 2\,, \quad P_2 = 10 \quad \text{or} \quad 23 = Z \ Y$$
$$u_2 = 4\,, \quad P_2 = 7 \quad \text{or} \quad 20 = F \ T\,, \quad \text{etc.}$$

Hence the (c) scale appears as

$$\begin{matrix} D & Y & F & \\ & & & \cdots \\ N & Z & T & \end{matrix}$$

scale (d), determined as scale (b), using $v_2 \equiv 17P_2$,

$$v_2 = 0\,, \quad P_2 = 0 = D$$
$$v_2 = 1\,, \quad P_2 = 23 = Y$$
$$v_2 = 2\,, \quad P_2 = 20 = T\,, \quad \text{etc.}$$

Lower fixed part:

scale (e) formed from $v_1 \equiv 16P_1$, as in scale (c):

$$\begin{matrix} v_1: & 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\ P_1: & \begin{matrix} D \\ N \end{matrix} & & \begin{matrix} Q \\ W \end{matrix} & & \begin{matrix} Y \\ Z \end{matrix} & & \cdots \end{matrix}$$

The completed scales would have the appearance:

```
(a) D R B U E W G F X H Z K O N M L P I Q A T C V Y J S D R B ··· Y J S
(b) D U G H O L Q C J R E F Z N P A V S B W X K M I T Y D U G ··· I T Y

    D   Y   F   E   M   J   C   Q   B   O   H   A   P   D   Y  ···  P
(c) N   Z   T   I   R   K   X   W   L   S   V   G   U   N   Z       U
(d) D Y T I M K X W B S V A P N Z F E R J C Q L O H G U D Y T ··· H G U

    D   Q   Y   B   F   O   E   H   M   A   J   P   C   D   Q  ···  C
(e) N   W   Z   L   T   S   I   V   R   G   K   U   X   N   W       X
```

To encipher the pair P_1P_2 requires two settings, one to find C_1 and one to find C_2.

To find C_1*:* Set index of (c)-scale under P_1 of (b)-scale. Locate P_2 on (c) scale, and read C_1 on (a)-scale opposite P_2 on (c)-scale.

To find C_2*:* Set index of (c)-scale above P_1 of (e)-scale. Read C_2 on (a)-scale opposite P_2 on (d)-scale.

Settings to encipher $P_1P_2 = E\ R$:

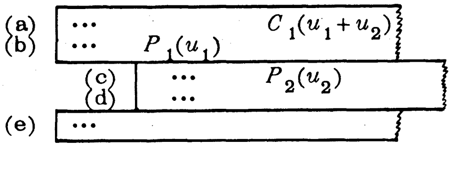
```
(a) D R B U E W G F X H Z K O N M L P I Q A T C ···
(b) D U G H O L Q C J R E F Z N P A V S B W X K ···
                        D   Y   F   E   M   J   ···
                    (c) N   Z   T   I   R   K
                    (d) D Y T I M K X W B S V A ···

    D   Q   Y   B   F   O   E   H   M   A   J   ···
(e) N   W   Z   L   T   S   I   V   R   G   K
```

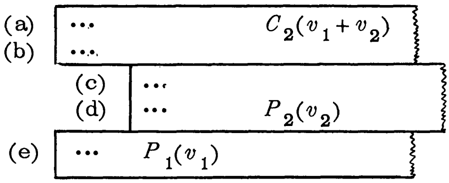
$P_1 = E$ on (b)-scale, $P_2 = R$ on (c)-scale; $C_1 = Q$ on (a)-scale.

```
                            (a) D R B U E W G F X H Z K O N M L ···
                            (b) D U G H O L Q C J R E F Z N P A ···
    D   Y   F   E   M   J   C   Q   B   O   H   A   P   D   Y   ···
(c) N   Z   T   I   R   K   X   W   L   S   V   Q   U   N   Z
(d) D Y T I M K X W B S V A P N Z F E R J C Q L O H G U D Y T I ···
                                D   Q   Y   B   F   O   E   H   ···
                            (e) N   W   Z   L   T   S   I   V
```

$P_1 = E$ on (e)-scale, $P_2 = R$ on (d)-scale, $C_2 = U$ on (a)-scale. (Index of (c)-scale is at $\frac{D}{N}$). In the above setting to find C_2 we have used the (c)-scale index in the middle of the scale.

The general settings are indicated by the diagrams below:





A few moments study of the construction of the rule as given above with reference to (1.4) will show how to construct the rule corresponding to the general case (1.3). Thus, scales (b), (c), (d), (e) are based respectively on the relations

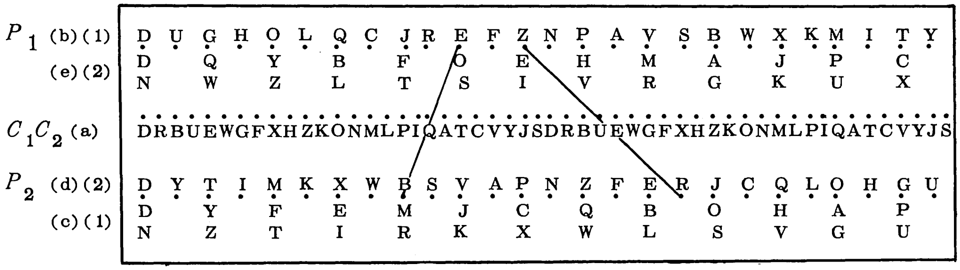
$$u_1 \equiv aP_1, \quad u_2 \equiv bP_2, \quad v_2 \equiv dP_2, \quad v_1 \equiv cP_1 .$$

Scale (a) is determined by whatever alphabetic permutation (1.1) is used.

The validity of the slide-rule encipherment is obvious from an inspection of the above general settings. To decipher, simply interchange the roles of P_1P_2 and C_1C_2 in the slide-rule operations, since the matrix of (1.3) is taken as involutory.

4. *Alignment Chart Method.* A chart with three parallel scales based on relations $C_1 = u_1+u_2$, $C_2 = v_1+v_2$ is constructed as follows. The inner scale is midway between the two outer scales and is the same as the (a)-scale of section 3. We call this now the C_1C_2 scale. The upper scale contains the (b) and (e) scales (P_1 scale), and the lower scale, the (d) and (c) scales (P_2 scale) of section 3. The modulus is 1 for the P_1 and P_2 scales, and ½ for the C_1C_2 scale (see [4], pp. 50-52).

The form of the chart for (1.4) is given below.


P_1 (b)(1)
(e)(2)
C_1C_2 (a)
P_2 (d)(2)
(c)(1)
D U G H O L Q C J R E F Z N P A V S B W X K M I T Y
D Q Y B F O E H M A J P C
N W Z L T S I V R G K U X
DRBUEWGFXHZKONMLPIQATCVYJSDRBUEWGFXHZKONMLPIQATCVYJS
D Y T I M K X W B S V A P N Z F E R J C Q L O H G U
D Y F E M J C Q B O H A P
N Z T I R K X W L S V G U

To encipher $P_1P_2 = ER$, determine letter $C_1 = Q$ on the middle (a) scale which is the intersection of the line joining $P_1 = E$ of (b) and $P_2 = R$ of (c) scales with the (a) scale. Then determine $C_2 = U$ as the intersection of the line joining $P_1 = E$ of (e) scale and $P_2 = R$ of (d) scale with the (a) scale. For deciphering the same chart and procedure is used as for enciphering.

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A THEOREM ON DETERMINANTS

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Theorem : If the differences of each pair of corresponding elements of any two columns (or rows) of a determinant are equal, then any quantity may be *added* to each element of the determinant without changing its value.

Without loss of generality the differences of the corresponding elements of the first two columns may be considered to be the equal ones. Then

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12}-a_{11} & \cdots & a_{1n}-a_{11} \\ a_{21} & a_{22}-a_{21} & \cdots & a_{2n}-a_{21} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2}-a_{n1} & \cdots & a_{nn}-a_{n1} \end{vmatrix} + \begin{vmatrix} b & a_{12}-a_{11} & \cdots & a_{1n}-a_{11} \\ b & a_{22}-a_{21} & \cdots & a_{2n}-a_{21} \\ \cdot & \cdot & \cdots & \cdot \\ b & a_{n2}-a_{n1} & \cdots & a_{nn}-a_{n1} \end{vmatrix},$$

for the value of the last determinant is zero, since the corresponding elements of its first two columns are proportional. So, when the last two determinants are added, and then the elements of the first column are added to each of the other columns, we have

$$D = \begin{vmatrix} a_{11}+b & a_{12}+b & \cdots & a_{1n}+b \\ a_{21}+b & a_{22}+b & \cdots & a_{2n}+b \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1}+b & a_{n2}+b & \cdots & a_{nn}+b \end{vmatrix}.$$

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ON CONGRUENCE PROPERTIES OF LEGENDRE POLYNOMIALS

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1. Congruence properties of the classical orthogonal polynomials have received little attention. We cite here some well-known congruences of Legendre polynomials:

The congruence of Schur is

$$P_n \equiv P_{a_0} P^{p}_{a_1} P^{p^2}_{a_2} \cdots P^{p^n}_{a_n} \quad (\mathrm{mod}\ p)\ , \tag{1.1}$$

where p is an odd prime and

$$n = a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n \quad (0 \leqq a_i < p)$$

The congruences of Carlitz [1] are

$$2^p P_p(x) \equiv \{(x-1)^p + (x+1)^p\} \quad (\mathrm{mod}\ p^2)\ , \tag{1.2}$$

and

$$2^{2p}\{P_{2p}(x) - P_p^2(x)\} \equiv 2(x^2-1)^p \quad (\mathrm{mod}\ p^2)\ . \tag{1.3}$$

The congruence of Wahab [2] is

$$P_{kp}(x) \equiv P_k^p(x) \quad (\mathrm{mod}\ p)\ . \tag{1.4}$$

The object of this present paper is to study some new congruence properties of Legendre polynomials.

2. Before studying our desired congruence properties of Legendre polynomials, we shall mention some congruence properties of certain particular binomial coefficients. In [1, p. 481] we have

$$\binom{2p}{p} = 2\binom{2p-1}{p-1} \equiv 2 \quad (\mathrm{mod}\ p^2)\ . \tag{2.1}$$

This congruence was proved by several mathematicians [3]. Again it was suggested by David Segal [4] that the congruence

$$\binom{2p-1}{p-1} \equiv 1 \quad (\mathrm{mod}\ p^2)$$

is a necessary and sufficient condition for p to be an odd prime. Also in [2, p. 174] we notice that

$$\binom{kp}{j} \equiv 0 \quad (\mathrm{mod}\ p)\ ,\ j \neq ip \quad \text{and} \quad \binom{kp}{ip} \equiv \binom{k}{i} \quad (\mathrm{mod}\ p)\ . \tag{2.2}$$

Now we know that [5]

$$\binom{kp}{ip} \equiv \binom{k}{i} \quad (\mathrm{mod}\ p^2)\ ,\ p > 2,\ i = 1,\ 2,\ \cdots,\ (k-1)\ . \tag{2.3}$$

Although this congruence viz., (2.3) is included in a more general result of Glashier [5, p. 111], yet we like to present our independent method of proof of (2.3), since a particular form of Glashier's congruence has recently been proved by several mathematicians [3, pp. 590-92].

To prove (2.3), we require Vandermonde's theorem

$$\begin{aligned}(2.4)\qquad \binom{m+n}{p} &= \binom{m}{p} + \binom{n}{1}\binom{m}{p-1} + \binom{n}{2}\binom{m}{p-2} + \cdots + \binom{n}{p-1}\binom{m}{1} + \binom{n}{p} \\ &= \sum_{i=0}^{p} \binom{n}{i}\binom{m}{p-i}\,; \quad p \not> m \text{ or } n\,.\end{aligned}$$

We first notice that

$$\binom{3p}{p} = \binom{2p+p}{p} = \binom{2p}{p} + \binom{p}{1}\binom{2p}{p-1} + \binom{p}{2}\binom{2p}{p-2} + \cdots + \binom{p}{p-1}\binom{2p}{1} + \binom{2p}{0}\,.$$

It follows therefore from (2.1) and (2.2) that

$$(2.5)\qquad \binom{3p}{p} \equiv \binom{2}{1} + 1 = \binom{3}{1} \quad (\text{mod}\cdot p^2)\,.$$

Next we observe that

$$\binom{4p}{p} = \binom{3p+p}{p} = \binom{3p}{p} + \binom{p}{1}\binom{3p}{p-1} + \binom{p}{2}\binom{3p}{p-2} + \cdots + \binom{p}{p-1}\binom{3p}{1} + \binom{3p}{0}\,.$$

Thus it follows from (2.2) and (2.5) that

$$(2.6)\qquad \binom{4p}{p} \equiv \binom{3}{1} + 1 = \binom{4}{1} \quad (\text{mod } p^2)\,.$$

Again,

$$\binom{4p}{2p} = \binom{2p+2p}{2p} = \binom{2p}{2p} + \binom{2p}{1}\binom{2p}{2p-1} + \binom{2p}{2}\binom{2p}{2p-2} + \cdots + \binom{2p}{2p-1}\binom{2p}{1} + \binom{2p}{0}\,.$$

It follows therefore from (2.1) and (2.2) that

$$(2.7)\qquad \binom{4p}{2p} \equiv \binom{2p}{2p} + \binom{2p}{p}\binom{2p}{2p-p} + \binom{2p}{0} \equiv 2 + \binom{2}{1}^2 = \binom{4}{2} \quad (\text{mod } p^2)\,.$$

In like manner we see that (2.3) is true for $k = 5, 6$, etc. Now let us assume that (2.3) is true for $1 < k \leqq n$. We shall now prove that

$$\binom{(n+1)p}{ip} \equiv \binom{n+1}{i} \quad (\text{mod } p^2),\ i = 1, 2, \cdots, n\,.$$

It may be noted that we need not prove this congruence for $i = 1, 2, \cdots, n$, since $\binom{n}{p} = \binom{n}{n-p}$. It is sufficient to prove it for $i < \frac{n+1}{2}$ or $< [\frac{n+1}{2}]$, according as n is odd or even. First we have

$$\binom{(n+1)p}{p} = \binom{np}{p} + \binom{p}{1}\binom{np}{p-1} + \cdots + \binom{p}{p-1}\binom{np}{1} + \binom{np}{0}\,,$$

$$(2.8)\qquad \therefore\ \binom{(n+1)p}{p} \equiv \binom{n}{1} + 1 = \binom{n+1}{1} \quad (\text{mod } p^2)\,.$$

Secondly we have

$$\binom{(n+1)p}{2p} = \binom{(n-1)p}{2p} + \binom{2p}{1}\binom{(n-1)p}{2p-1} + \cdots + \binom{2p}{2p-1}\binom{(n-1)p}{1} + \binom{(n-1)p}{0},$$

$$(2.9) \qquad \therefore \ \binom{(n+1)p}{2p} \equiv \binom{n-1}{2} + \binom{2}{1}\binom{n-1}{1} + 1 = \binom{n+1}{2} \pmod{p^2}.$$

Lastly, for any integer $m < \frac{n+1}{2}$, we have

$$\binom{(n+1)p}{mp} = \binom{(n-m+1)p+mp}{mp} = \sum_{i=0}^{mp} \binom{mp}{i}\binom{(n-m+1)p}{mp-i},$$

$$(2.10) \ \therefore \ \binom{(n+1)p}{mp} \equiv \sum_{i=0}^{m} \binom{m}{i}\binom{n-m+1}{m-i} = \binom{(n-m+1)+m}{m} = \binom{n+1}{m} \pmod{p^2}.$$

This completes the proof of (2.3).

3. We shall now study some congruence properties of Legendre polynomials. It is well known [1, p. 481] that the Legendre polynomials of degree n can be written in the form:

$$(3.1) \qquad 2^n P_n(x) = \sum_{s=0}^{n} \binom{n}{s}^2 (x-1)^s (x+1)^{n-s}.$$

From (3.1) it is evident that

$$2^{3p} P_{3p}(x) \equiv (x-1)^{3p} + \binom{3p}{p}^2 (x-1)^p (x+1)^{2p} + \binom{3p}{2p}^2 (x-1)^{2p}(x+1)^p + (x+1)^{3p} \pmod{p^2}.$$

Now, since

$$\binom{3p}{2p} = \binom{3p}{p} \equiv \binom{3}{1} \pmod{p^2},$$

we have

$$(3.2) \ 2^{3p} P_{3p}(x) \equiv (x-1)^{3p} + 9(x^2-1)^p\{(x-1)^p + (x+1)^p\} + (x+1)^{3p} \pmod{p^2}.$$

But from (1.2) we obtain

$$(3.3) \ 2^{3p} P_p^3(x) \equiv (x-1)^{3p} + 3(x^2-1)^p\{(x-1)^p + (x+1)^p\} + (x+1)^{3p} \pmod{p^2}.$$

From (3.2) and (3.3), it follows therefore

$$(3.4) \qquad 2^{3p}[P_{3p}(x) - P_p^3(x)] \equiv 6(x^2-1)^p\{(x-1)^p + (x+1)^p\} \pmod{p^2}.$$

Again, from (1.2), (1.3) and (3.4) we get

$$(3.5) \qquad P_{3p}(x) \equiv P_p^3(x) + 3\{P_{2p}(x) - P_p^2(x)\}P_p(x) \pmod{p^2}$$

The congruence (3.5) for $P_{3p}(x)$ suggests the possibility of a like result for $P_{kp}(x)$ generally. Indeed, since

$$(3.6)\qquad 2^{4p}P_{4p}(x) \equiv (x-1)^{4p}+16(x-1)^{3p}(x+1)^{p}+36(x^2-1)^{2p} + 16(x-1)^{p}(x+1)^{3p}+(x+1)^{4p} \pmod{p^2}$$

and

$$(3.7)\qquad 2^{4p}P_p^4(x) \equiv (x-1)^{4p}+4(x-1)^{3p}(x+1)^{p}+6(x^2-1)^{2p} + 4(x-1)^{p}(x+1)^{3p}+(x+1)^{4p} \pmod{p^2}\ ,$$

we get

$$2^{4p}[P_{4p}(x)-P_p^4(x)] \equiv 12(x^2-1)^p\{(x-1)^{2p}+(x+1)^{2p}\}+30(x^2-1)^{2p} \pmod{p^2}$$

$$(3.8)\qquad \equiv 12(x^2-1)^p\{(x-1)^{p}+(x+1)^{p}\}^2+6(x^2-1)^{2p} \pmod{p^2}\ .$$

Comparing (1.2), (1.3) and (3.8) we get

$$(3.9)\quad P_{4p}(x) \equiv P_p^4(x)+6\{P_{2p}(x)-P_p^2(x)\}P_p^2(x)+\frac{3}{2}\{P_{2p}(x)-P_p^2(x)\}^2 \pmod{p^2}.$$

4. In general, we have from (3.1)

$$(4.1)\quad 2^{kp}P_{kp}(x) \equiv (x+1)^{kp}+(x-1)^{kp}+\sum_{i=1}^{k-1}\binom{k}{i}^2(x-1)^{ip}(x+1)^{(k-i)p} \pmod{p^2}.$$

But from (1.2) we obtain

$$(4.2)\quad 2^{kp}P_p^k(x) \equiv (x+1)^{kp}+(x-1)^{kp}+\sum_{i=1}^{k-1}\binom{k}{i}(x-1)^{ip}(x+1)^{(k-i)p} \pmod{p^2}.$$

Thus it follows from (4.1) and (4.2) that

$$(4.3)\quad 2^{kp}[P_{kp}(x)-P_p^k(x)] \equiv \sum_{i=1}^{k-1}\binom{k}{i}\left\{\binom{k}{i}-1\right\}(x-1)^{ip}(x+1)^{(k-i)p} \pmod{p^2}.$$

Now first suppose k to be odd, so that we put $k = 2m+1$. ∴ from (4.3) we obtain

$$2^{(2m+1)p}[P_{(2m+1)p}-P_p^{(2m+1)}]$$

$$\equiv \sum_{i=1}^{2m}\binom{2m+1}{i}\left\{\binom{2m+1}{i}-1\right\}(x-1)^{ip}(x+1)^{(2m+1-i)p} \pmod{p^2}$$

$$(4.4)\quad \equiv \sum_{i=1}^{m}\binom{2m+1}{i}\left\{\binom{2m+1}{i}-1\right\}(x^2-1)^{ip}\{(x-1)^{(2m+1-2i)p}+(x+1)^{(2m+1-2i)p}\}.$$

Now we like to show that there exist suitable constants $c_i (i = 1, 2, \cdots, m)$, such that

$$(4.5)\quad P_{(2m+1)p} \equiv P_p^{(2m+1)} + \sum_{i=1}^{m} c_i(P_{2p}-P_p^2)^i P_p^{(2m+1-2i)} \pmod{p^2},$$

$$(4.6)\quad \text{i. e.,}\ P_{(2m+1)p} - P_p^{(2m+1)} \equiv \sum_{i=1}^{m} c_i(P_{2p}-P_p^2)^i P_p^{(2m+1-2i)} \pmod{p^2}.$$

From (1.2), (1.3) and (4.6) we get

$$2^{(2m+1)p}[P_{(2m+1)p} - P_p^{(2m+1)}]$$

$$\equiv \sum_{i=1}^{m} 2^i c_i (x^2-1)^{ip}\{(x-1)^p + (x+1)^p\}^{(2m+1-2i)} \pmod{p^2}$$

$$(4.7)\qquad = \sum_{i=1}^{m} d_i (x^2-1)^{ip}\{(x-1)^{(2m+1-2i)p} + (x+1)^{(2m+1-2i)p}\},$$

where

$$(4.8)\quad \begin{aligned} d_1 &= 2c_1 \\ d_2 &= 2^2C_2 + 2c_1\binom{2m-1}{1} \\ d_3 &= 2^3c_3 + 2^2c_2\binom{2m-3}{1} + 2c_1\binom{2m-1}{2} \\ d_4 &= 2^4c_4 + 2^3c_3\binom{2m-5}{1} + 2^2c_2\binom{2m-3}{2} + 2c_1\binom{2m-1}{3} \\ &\cdots \\ d_m &= 2^m c_m + 2^{m-1}c_{m-1}\binom{3}{1} + 2^{m-2}c_{m-2}\binom{5}{2} + \cdots + 2c_1\binom{2m-1}{m-1}. \end{aligned}$$

Comparing (4.4), (4.7) and (4.8), we may thus state

Theorem 1. The Legendre polynomial $P_{(2m+1)p}(x)$ satisfies

$$P_{(2m+1)p} \equiv P_p^{(2m+1)} + c_1(P_{2p}-P_p^2)P_p^{(2m-1)} + c_2(P_{2p}-P_p^2)^2P_p^{(2m-3)}$$
$$+ \cdots + c_m(P_{2p}-P_p^2)^m P_p \pmod{p^2};$$

for all $m \geq 1$ and for any odd prime p, and where the constants c_i ($i = 1, 2, \cdots, m$), are given by the following equations:

$$\binom{2m+1}{1}\left\{\binom{2m+1}{1} - 1\right\} = 2c_1$$

$$\binom{2m+1}{2}\left\{\binom{2m+1}{2} - 1\right\} = 2^2c_2 + 2c_1\binom{2m-1}{1}$$

$$\binom{2m+1}{3}\left\{\binom{2m+1}{3}-1\right\} = 2^3c_3+2^2c_2\binom{2m-3}{1}+2c_1\binom{2m-1}{2}$$

$$\binom{2m+1}{4}\left\{\binom{2m+1}{4}-1\right\} = 2^4c_4+2^3c_3\binom{2m-5}{1}+2^2c_2\binom{2m-3}{2}+2c_1\binom{2m-1}{3}$$

$$\cdots$$

$$\binom{2m+1}{m}\left\{\binom{2m+1}{m}-1\right\} = 2^mc_m+2^{m-1}c_{m-1}\binom{3}{1}+2^{m-2}c_{m-2}\binom{5}{2}+\cdots+2c_1\binom{2m-1}{m-1}.$$

Next suppose k to be even, so that we now put $k = 2m$. $\therefore$ from (4.3) we have

$$2^{2mp}[P_{2mp}-P_p^{2m}]$$

$$\equiv \sum_{i=1}^{2m-1} \binom{2m}{i}\left\{\binom{2m}{i}-1\right\}(x-1)^{ip}(x+1)^{(2m-i)p} \pmod{p^2}$$

$$(4.9) \qquad = \sum_{i=1}^{m-1} \binom{2m}{i}\left\{\binom{2m}{i}-1\right\}(x^2-1)^{ip}\{(x-1)^{(2m-2i)p}+(x+1)^{(2m-2i)p}\}$$

$$+ \binom{2m}{m}\left\{\binom{2m}{m}-1\right\}(x^2-1)^{mp}.$$

Now we intend to show that there exist suitable constants c_i ($i = 1, 2, \cdots, m$), such that

$$(4.10) \qquad P_{2mp} \equiv P_p^{2m} + \sum_{i=1}^{m} c_i(P_{2p}-P_p^2)^i P_p^{(2m-2i)} \pmod{p^2}.$$

From (1.2), (1.3) and (4.10) we get

$$2^{2mp}[P_{2mp}-P_p^{2m}]$$

$$\equiv \sum_{i=1}^{m} 2^i c_i(x^2-1)^{ip}\{(x-1)^p+(x+1)^p\}^{(2m-2i)} \pmod{p^2}$$

$$= 2^m c_m(x^2-1)^{mp} + \sum_{i=1}^{m-1} 2^i c_i(x^2-1)^{ip}\{(x-1)^p+(x+1)^p\}^{(2m-2i)}$$

$$(4.11) \qquad = 2^m c_m(x^2-1)^{mp} + \sum_{i=1}^{m-1} d_i(x^2-1)^{ip}\{(x-1)^{(2m-2i)p}+(x+1)^{(2m-2i)p}\},$$

where

$$
\begin{aligned}
&d_1 = 2c_1\\
&d_2 = 2^2c_2 + 2c_1\binom{2m-2}{1}\\
(4.12)\qquad &d_3 = 2^3c_3 + 2^2c_2\binom{2m-4}{1} + 2c_1\binom{2m-2}{2}\\
&\qquad\cdots\\
&d_{m-1} = 2^{m-1}c_{m-1} + 2^{m-2}c_{m-2}\binom{4}{1} + \cdots + 2c_1\binom{2m-2}{m-2}
\end{aligned}
$$

Comparing (4.9), (4.11) and (4.12) we may thus state

Theorem 2. The Legendre polynomial $P_{2mp}(x)$ satisfies

$$
\begin{aligned}
P_{2mp} \equiv P_p^{2m} &+ c_1(P_{2p} - P_p^2)P_p^{2m-2} + c_2(P_{2p} - P_p^2)^2P_p^{2m-4}\\
&+ \cdots + c_m(P_{2p} - P_p^2)^m \pmod{p^2}\ ;
\end{aligned}
$$

for all $m \geqq 2$ and for any odd prime p, and where the constants c_i $(i = 1, 2, \cdots, m)$, are given by the following equations:

$$
\begin{aligned}
&\binom{2m}{1}\left\{\binom{2m}{1} - 1\right\} = 2c_1\\
&\binom{2m}{2}\left\{\binom{2m}{2} - 1\right\} = 2^2c_2 + 2c_1\binom{2m-2}{1}\\
&\binom{2m}{3}\left\{\binom{2m}{3} - 1\right\} = 2^3c_3 + 2^2c_2\binom{2m-4}{1} + 2c_1\binom{2m-2}{2}\\
&\qquad\cdots\\
&\binom{2m}{m-1}\left\{\binom{2m}{m-1} - 1\right\} = 2^{m-1}c_{m-1} + 2^{m-2}c_{m-2}\binom{4}{1} + \cdots + 2c_1\binom{2m-2}{m-2}\\
&\binom{2m}{m}\left\{\binom{2m}{m} - 1\right\} = 2^m c_m\ .
\end{aligned}
$$

Combining Theorem 1 and Theorem 2, we can remark that the Legendre polynomial $P_{kp}(x)$ satisfies

$$
(4.13)\qquad P_{kp}(x) - P_p^k(x) \equiv \sum_{i=1}^{[k/2]} c_i(P_{2p} - P_p^2)^iP_p^{(k-2i)}(x) \pmod{p^2}\ ,
$$

where $k = 3, 4, \cdots,$ and p is any odd prime and c_i are known constants.

5. Inverse Formula. From (3.5) we easily observe that

$$
(5.1)\qquad P_p^3(x) \equiv P_{3p}(x) - 3\{P_{2p}(x) - P_p^2(x)\}P_p(x) \pmod{p^2}\ .
$$

Again from (3.9) we have

$$
P_p^4(x) \equiv P_{4p}(x) - 6\{P_{2p}(x) - P_p^2(x)\}P_p^2(x) - \tfrac{3}{2}\{P_{2p}(x) - P_p^2(x)\}^2 \pmod{p^2}\ .
$$

Now, since

$$2^{1-2p}(x^2-1)^p \equiv P_{2p}(x) - P_p^2(x) \pmod{p^2},$$

we obtain

$$\begin{aligned} P_p^4(x) &\equiv P_{4p}(x) + 6\{P_{2p}(x) - P_p^2(x)\}\{2^{1-2p}(x^2-1)^p - P_{2p}(x)\} \\ &\quad - \frac{3}{2}\{P_{2p}(x) - P_p^2(x)\}^2 \pmod{p^2} \\ &\equiv P_{4p}(x) - 6\{P_{2p}(x) - P_p^2(x)\}P_{2p}(x) + 6\{P_{2p}(x) - P_p^2(x)\}^2 \\ &\quad - \frac{3}{2}\{P_{2p}(x) - P_p^2(x)\}^2 \pmod{p^2} \end{aligned}$$

$$\equiv P_{4p}(x) - 6\{P_{2p}(x) - P_p^2(x)\}P_{2p}(x) + \frac{9}{2}\{P_{2p}(x) - P_p^2(x)\}^2 \pmod{p^2}. \tag{5.2}$$

The congruences (5.1) and (5.2) for $P_p^3(x)$ and $P_p^4(x)$ respectively, suggest the possibility of a like result for $P_p^k(x)$ generally. Actually from (4.13) we have

$$P_p^k(x) \equiv P_{kp}(x) - \sum_{i=1}^{[k/2]} c_i(P_{2p} - P_p^2)^i P_p^{(k-2i)}(x) \pmod{p^2}. \tag{5.3}$$

By repeated application of (5.3), it would be possible to set up the following inverse formula

$$P_p^k(x) \equiv P_{kp}(x) + \sum_{i=1}^{[k/2]} d_i(P_{2p} - P_p^2)^i P_{(k-2i)p}(x) \pmod{p^2} \tag{5.4}$$

where the constants d_i can, no doubt, be explicitly determined; in particular $d_i = -\frac{1}{2}k(k-1)$.

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USE OF MATRICES FOR STUDY OF PLANE SECTIONS OF A QUADRIC

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Matrices can be used to deal with plane sections of a quadric in such a way that it is unnecessary to calculate all the elements of the transform matrix.

Suppose we are concerned with a quadric reduced to central form:

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy - 1 = 0 \ .$$

If it is not possible to choose the intersecting plane as one of the coordinate planes, it will be necessary to take a new triad of orthogonal axes OX, OY, OZ whose direction cosines are:

$$(l_1 \ \ m_1 \ \ n_1) \quad (l_2 \ \ m_2 \ \ n_2) \quad (l_3 \ \ m_3 \ \ n_3)$$

referred to the original system of axes. We take $l_3 m_3 n_3$ normal to the intersecting plane

$$\lambda x + \mu y + \nu z = 0 \qquad \text{I}$$

Let Q be the matrix of the original quadratic form:

$$Q = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}.$$

The transformed matrix will be Q' where $Q' = TQT'$, T being

$$\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$

Having identified $l_3 m_3 n_3$ with the normal to *I* we still possess one degree of freedom in the choice of $(l_1 m_1 n_1)$ $(l_2 m_2 n_2)$ and this degree of freedom may be employed to reduce the algebra or take advantage of some geometrical feature. Of the nine elements of the transformed matrix we are only concerned with the four in the top left hand corner as we are going to put z equal to zero to study the plane section. We need not calculate the remaining five elements.

As a practical example suppose we wish to find the product of the lengths of the axes of the section of the ellipsoid

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} - 1 = 0$$

by the plane *I*. This is a well-known problem — see W. H. Macaulay *Solid*

Geometry, Cambridge University Press, 1930, page 90. The method there adopted is to make the plane l touch the cone

$$\left(\frac{1}{A^2}-\frac{1}{r^2}\right)x^2+\left(\frac{1}{B^2}-\frac{1}{r^2}\right)y^2+\left(\frac{1}{C^2}-\frac{1}{r^2}\right)z^2=0\,.$$

This yields a biquadratic in r from which

$$r_1^2\cdot r_2^2=\frac{A^2\cdot B^2\cdot C^2}{A^2\lambda^2+B^2\mu^2+C^2\nu^2}\,.$$

By our method we make $l_3=\lambda$, $m_3=\mu$, $n_3=\nu$ while to absorb the one degree of freedom mentioned we take OY as the line of intersection of Oxy and the plane l. Finally we take OX perpendicular to OY and OZ. Thus the new triad of axes has the following direction cosines with respect to the original axes:

$$II\ \begin{cases} l_1=\dfrac{-\lambda\nu}{\sqrt{\lambda^2+\mu^2}} & m_1=\dfrac{-\mu\nu}{\sqrt{\lambda^2+\mu^2}} \qquad n_1=\sqrt{\lambda^2+\mu^2}\\[2ex] l_2=\dfrac{\mu}{\sqrt{\lambda^2+\mu^2}} & m_2=\dfrac{-\lambda}{\sqrt{\lambda^2+\mu^2}} \qquad n_2=0\\[2ex] l_3=\lambda & m_3=\mu \qquad\qquad n_3=\nu \end{cases}\,.$$

The four relevant elements of the transformed matrix are:

$$\begin{bmatrix} \dfrac{l_1^2}{A^2}+\dfrac{m_1^2}{B^2}+\dfrac{n_1^2}{C^2} & \dfrac{l_1l_2}{A^2}+\dfrac{m_1m_2}{B^2}+\dfrac{n_1n_2}{C^2} & \cdots\\[2ex] \dfrac{l_1l_2}{A^2}+\dfrac{m_1m_2}{B^2}+\dfrac{n_1n_2}{C^2} & \dfrac{l_2^2}{A^2}+\dfrac{m_2^2}{B^2}+\dfrac{n_2^2}{C^2} & \cdots\\[2ex] \cdots & \cdots & \cdots \end{bmatrix}\,.$$

The border elements being left blank as no need exists to calculate them. Hence the section by plane l is the ellipse:

$$X^2\left(\frac{l_1^2}{A^2}+\frac{m_1^2}{B^2}+\frac{n_1^2}{C^2}\right)+2XY\left(\frac{l_1l_2}{A^2}+\frac{m_1m_2}{B^2}+\frac{n_1n_2}{C^2}\right)+Y^2\left(\frac{l_2^2}{A^2}+\frac{m_2^2}{B^2}+\frac{n_2^2}{C^2}\right)=1\,.$$

For the lengths of the axes of such an ellipse see C. Smith *Conic Sections,* Macmillan, 1921, page 230. If r is the length of an axis:

$$\frac{1}{r^4}-\frac{k}{r^2}+\left(\frac{l_1^2}{A^2}+\frac{m_1^2}{B^2}+\frac{n_1^2}{C^2}\right)\left(\frac{l_2^2}{A^2}+\frac{m_2^2}{B^2}+\frac{n_2^2}{C^2}\right)-\left(\frac{l_1l_2}{A^2}+\frac{m_1m_2}{B^2}+\frac{n_1n_2}{C^2}\right)^2=0\,.$$

where k is irrelevant as we are only interested in the product of the roots in r. Thus:

$$\frac{1}{r_1^2\cdot r_2^2}=\frac{l_1^2m_2^2+l_2^2m_1^2-2l_1l_2m_1m_2}{A^2B^2}+2\text{ symmetrical terms}\,.$$

On substituting the values of the direction cosines from *II* we have:

$$r_1^2 \cdot r_2^2 = \frac{A^2B^2C^2}{A^2\lambda^2+B^2\mu^2+C^2\nu^2}$$

as given by Macaulay.

There are occasions when it is necessary to calculate only two of the nine elements of the transformed matrix. Such an example is given on page 170 by Macaulay, to prove that the tangent planes of the cone:

$$\frac{x^2}{b+c}+\frac{y^2}{c+g}+\frac{z^2}{g+b} = 0 - III$$

cut the quadric

$$ax^2+bz^2+cz^2 = 1 - IV$$

in rectangular hyperbolas.

Let $\lambda x+\mu y+\nu z = 0$ be a tangent plane of the cone, then $\lambda\,\mu\,\nu$ satisfy

$$\lambda^2(b+c)+\mu^2(c+a)+\nu^2(a+b) = 0 - V\ ,$$

this being the known tangential equation of the quadric *III*. We should note that the symbols a b c are distinct from those used in the original quadratic form for S. We now transform to the same set of orthogonal axes as those used previously, i.e. the plane OXY is that of the intersecting plane $\lambda x+\mu y+\nu z = 0$, OY is the line of intersection of $\lambda x+\mu y+\nu z = 0$ with Oxy. Finally OX is perpendicular to OY and OZ. Hence for the new triad of axes we have the same direction cosines as set out in *II* above. Also for the conic of intersection of $\lambda x+\mu y+\nu z = 0$ with the quadric *IV* we have the four relevant elements of the transformed matrix:

$$T\cdot\begin{bmatrix} a & \cdot & \cdot \\ \cdot & b & \cdot \\ \cdot & \cdot & c \end{bmatrix}\cdot T' = \begin{bmatrix} al_1^2+bm_1^2+cn_1^2 & al_1l_2+bm_1m_2+cn_1n_2 & \cdots \\ al_1l_2+bm_1m_2+cn_1n_2 & al_2^2+bm_2^2+cn_2^2 & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} - VI\ .$$

The condition that the resulting equation in X and Y should represent a rectangular hyperbola in the plane $Z = 0$ is that the sum of the coefficients of X^2 and Y^2 should be zero: that is, that the sum of the top two diagonal elements of *VI* should be zero. Substituting for the direction cosines from *II* we have

$$a(\lambda^2\nu^2+\mu^2)+b(\mu^2\nu^2+\lambda^2)+c(\lambda^2+\mu^2)^2 = 0$$

as the sufficient condition for the conic of intersection to be a rectangular hyperbola: and this reduces to V which is already satisfied as the condition that the plane $\lambda x+\mu y+\nu z = 0$ touches the cone *III*.

It is considered that this method should prove convenient in the solution of other similar intersection problems.

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TEACHING OF MATHEMATICS

Edited by

Rothwell Stephens

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to *Rothwell Stephens Mathematics Department, Knox College, Galesburg, Illinois.*

THE SCHNITZELBANK SCHOOL OF MATHEMATICAL PEDAGOGY

(A How-to-do-it Manual)

Ya. I. M. Fedëp

Introduction. A primary concern of many departments of mathematics is the training of engineers in the mathematical arts and sciences, both at the undergraduate and graduate levels. The present essay is designed to enlighten our readers concerning the only truly modern and efficient method of doing the job.

1. The first thing to do is to obtain a textbook for the course under consideration, and the method of accomplishing this will now be described. Before having luncheon with the publisher's representative, be sure to tell him that you are looking for a new textbook, especially designed to fit the needs of the student of engineering in the modern world. This always results in a freelunch. After several months of using this procedure, you will have a good idea of what is available.

In due time the samples of the various publishing houses will arrive in the mathematics office (total weight of samples: c. one ton). In the meantime it is necessary to carry out the following steps. (1) Obtain a scale which will weigh objects between five (5) and fifteen (15) pounds.[1] (2) Instruct the secretary in the use of this instrument. As each textbook comes in the mail, she is to unwrap it, throw away the wrappings (which should not be allowed to accumulate on account of the fire hazard), weigh it, and record the weight in her log book. After all samples have arrived, she is to make a bar graph displaying the results, together with a report of this survey, to be placed on your desk.[2] Then you are in a position to select the appropriate textbook. One tries, of course, to maximize the weight. In case there is no unique solution to the problem in this form, other properties of the material may be used such as color, personal density,[3] and the like.[4]

2. After the text has been selected in the modern, efficient manner described in section 1, it is necessary to prepare a syllabus. Again it is desirable to have an efficient, and perhaps even an ambitious secretary

for this purpose. If by some chance you are blessed with such a jewel, your next task will indeed be easy. Let her write the syllabus. There is one note of caution here, however. If she should succeed in doing *too* well at such a chore, it might indicate that she has wised up to the academic mores so much that she will get delusions of grandeur. There have been cases in which secretaries have gone behind the professor's back and submitted secret copies of such work to other departments as a dissertation, thereupon obtaining a Ph. D. degree. In certain subcases the overly ambitious girl has even, in time, taken over the professor's job. This is potentially a difficult problem. If you feel that your secretary is about to embark on such a path, there is only one sure counter-move. Marry her. If she still wants your job after that, well, the solution is obvious.

3. Having selected the textbook and prepared the syllabus, the only remaining task is to teach the course. The modern, efficient method here is due to the renowned Schnitzelbank.[5] His scheme is absolutely universal, and may be used on students of any age or condition. This eminent one has taught all subjects at all levels by this method with equally good results. His pupils are legion.

Like all good modern tools for instruction, its basis is audio-visual. In fact, any projection machine may be used for the purpose. However, when the author first encountered the method at the hands of the master at a certain institution in Indianapolis in 1938, the hardware was still in a rather crude state. The audience, on the other hand, included some rather well-known mathematicians. As soon as the present author entered the room, he immediately caught the spirit of the occasion. This was unusual at such a symposium in those days. For, it is well known that most mathematicians believe that the only useful audio-visual equipment consists of (a) their own vocal cords, and (b) blackboard and chalk. Nevertheless, at this particular gathering, everyone entered into the spirit of the thing, listened from time to time, and participated vigorously in the proceedings. Little did anyone there realise, until years later, what a marvelous lesson in the art of audio-visual education he was receiving. The whole thing was done with finesse. Surely if this method was so effective on blasé professional mathematicians, it ought to work wonders with our young prospective engineers.

Schnitzelbank's method (1938) will now be described in detail. For convenience, he will be denoted henceforth by "S".

Step 1. The instructor places a collection of charts on an easel, with the first chart, a picture of S, in view of the students. This picture is so unusual that it should provoke comment. With the help of a pointer,[6] the instructor displays the first chart, saying, "Is this not a S?" The students, if they have been brought to the proper state of readiness, respond, "Yes, this is a S."[8]

Step 2. The instructor flips, to the next chart. By complete induction, to be carried out by the reader, one can arrive at any desired chart (In S's original version, no. 13 was a lulu.)

4. Having presented the general historical and philosophical background, we now proceed to the application of S's method to the instruction of engineers and others lucky enough to be present in the classes under discussion.

Any good instructor will make sure that his students are "ready." This may be accomplished by relating suitable anecdotes, e. g. from the history of technology.[9] Next, through a suitably prepared syllabus (sec. 2) and by means of a weighty and expensive textbook (sec. 1), the instructor impresses the student with the fact that this course is going to be "plenty tough."

As soon as the students are in a steady state of readiness, the instructor begins his presentation in a manner exactly like that of Dr. S. The first image to be projected is ordinarily that of the instructor himself, together with his full name, degrees, institutions, any books he may have written, and the like.[10] Of course this first chart will have no effect on freshmen, who seldom have any interest in such matters. The second thing to be displayed is some suitable mathematical object such as a differential equation:

$$\frac{d^{17}y}{dx^{17}} - xy'y^{8} - \log\left(\frac{dy}{dx}\right) - \arcsin(x-y^{15}) = 0 .$$

By means of a pointer (see footnote 6) the instructor indicates this object and says in a loud voice, "THIS IS A DIFFERENTIAL EQUATION." The students then respond, en masse, "OH YEAH?" Before they can say anything else, he must quickly flip, to the next slide, a 13×21 matrix. This is handled in a similar manner.[11] Having indicated the first two steps of the process, we leave it to the reader to supply the remaining ones. These ought to provide a veritable "Cook's Tour" (adv.) through the resplendent rain forest of modern mathematics for the engineer.

5. In conclusion, we have outlined in some detail the way to be followed to do the job in a modern and efficient manner. Following the precepts so ably put forth by S (1938), with the appropriate modifications suggested in the present paper, you can succeed! There is absolutely no reason why your students should not get to the top of the ladder in modern industrial life.[12]

FOOTNOTES

1. This may be obtained on requisition. However, in this instance it may be wise, in the interest of efficiency, to revert to an older method, and secure it at your local emporium.

2. It has been found by experience that some secretaries are not keen on doing this kind of work. The solution of this problem is left to the reader, with one hint which has sometimes proved useful. In case the secretary is of the more intellectual type, it has been found feasible at certain institutions for her to use her report (after the addition of footnotes) as a Master's Thesis in some

suitable department.

3. The *personal density* of a book B with respect to a mathematician M may be defined as the quotient of the number of pages containing M's name divided by the total number of pages in B.

4. Of course the clever reader has undoubtedly thought of a much easier method involving a maximization of the price. This solution is, however, not to be recommended. For one thing, it is a little too obvious.

5. Any resemblance between this famous individual and a childish game of the same name is, of course, purely coincidental.

6. Cat. No. VBI 3141592...

7. This footnote got lost somewhere.

8. For the convenience of the reader, the questions and answers have been translated into English from the German originals.

9. The methodof establishing readiness which was used by S (1938) as described in sec. 3 is definitely *not* recommended for a class containing any students under twenty one (21) years of age (See the Penal Code, Alcoholic Beverages Division, of practically any state.)

10. This item may be omitted by the more modest type of instructor. However the omission should be carefully considered for its effect on morale.

11. Some old fashioned and still unenlightened teachers still insist on multiplying a couple of 2×2 matrices together at this point. Such a practice is clearly not in the spirit of S (1938).

12. If possible, however, it is preferable not to travel in aerospace vehicles designed by your students.

Bjerklos, S. W. T.

LEX AND INCLUDED ANGLE

As one grows wise he reads the lines
Of documents to which he signs.
And wiser still, avoids the flaw
Of carelessness with cosign law.

Marlow Sholander

AN ANALYTICAL METHOD FOR SOLVING BASIC INEQUALITIES

R. K. Coburn

During my twelve years of teaching, I have found that most of the texts covering inequalities approach the solution of the basic inequality problem from a graphical viewpoint. The purpose of this paper is to present an analytical approach to the solution of problems of this type.

The problems that readily lend themselves to solution by this analytical approach are of the following two basic types : $U \cdot V > 0$ and $U \cdot V < 0$. "U" and "V" are considered in this discussion to be linear functions of a single variable, say x, but this method can be extended to include factors of higher degree than the first. A few special examples where "U" and "V" are of the second degree in x are contained in the appendix to this paper.

Under the restrictions set forth on "U" and "V" above, every inequality of the form $U \cdot V > 0$ or $U \cdot V < 0$ can be put into one of the following forms using the basic laws of inequalities :

$(x+c)(x+d) > 0$ or $(x+c)(x+d) < 0$ (Where "c" and "d" are constants.)

Hereafter, this form will be referred to as the standard form.

THEOREM I: The solution to inequalities of the form $(x+c)(x+d) > 0$ is obtained by finding the range of values of x for which the smaller factor is positive and *the range of values of x for which the larger factor is negative.*

PROOF : From the algebraic value of the constants c and d, it is possible to determine which of the factors will have the greatest value and which the least value for any given value of x, e. g., if $c > d$ then $(x+c) > (x+d)$ for any given value of x and vice versa. For the product of two factors to be positive the factors must have the same sign, i. e., both factors must be positive or both factors must be negative for a given value of x. If we assume $c > d$, which takes nothing from the general nature of the problem, then if both factors are to be positive for a given value of x the smaller factor $(x+d)$ must be greater than zero. For if this is true then the factor $(x+c)$ will also be greater than zero since $(x+c)$ is greater than $(x+d)$ by hypothesis. Thus the inequality will be satisfied if $x > -d$. If both factors are to be negative for a given value of x then the larger factor $(x+c)$ must be less than zero; for if this is true then the factor $(x+d)$ will also be less than zero by hypothesis. So the inequality will also hold true for $x < -c$. Thus the inequality is satisfied if $x > -d$ or if $x < -c$. This same type of discussion will hold true for the quotient of two linear factors, e. g.

$$\frac{x+c}{x+d} > 0 \, .$$

THEOREM II: The solution to inequalities of the form $(x+c)(x+d) < 0$ is *obtained by finding the range of values of x for which the smaller factor is negative and the larger factor is positive.*

PROOF: Assume again that $c > d$ and thus $(x+c) > (x+d)$. Now for the product of two factors to be negative, the factors must have opposite signs, i.e. one factor must be positive for a given range of values of x while the other factor must be negative for the same range of values. Obviously, the largestof the two factors must be positive and the smallest negative, since any positive number is greater than any negative number. So the values of x that satisfy this inequality will be those values satisfying the inequalities $(x+c) > 0$ and $(x+d) < 0$, or $-c < x < -d$.

(An easy way to remember the results suggested in both theorems one and two is to note that in both cases the smaller factor of the product is given the same inequality sign as the inequality from which it was taken.)

It is interesting to note that the solution of inequalities of the type $(x+c)(x+d) > 0$, where $c > d$, leads to values of x lying in two open type intervals $x > -d$ and $x < -c$, while the solution of the inequalities of the type $(x+c)(x+d) < 0$ leads to values of x lying in one open interval $-c < x < -d$.

APPENDIX

In summarizing the salient points brought out in this short paper, a few problems are included in the appendix to illustrate the use of these theorems in solving some typical problems in the field of basic inequalities.

EXAMPLE I:

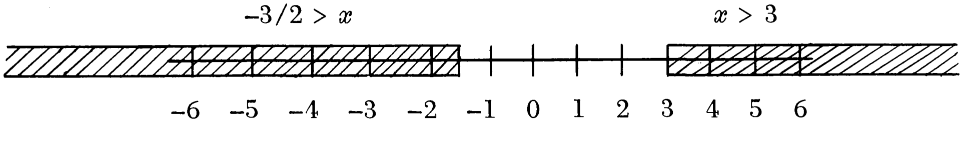
Solve the inequality

$$\frac{2x+3}{3x-9} > 0 .$$

Multiplying both sides of the inequality by $\frac{3}{2}$, we obtain $\dfrac{x+\frac{3}{2}}{x-3} > 0$. Since $(x+3/2) > (x-3)$ for all real values of x, using Theorem I, we obtain the following solution:

Both factors positive	*Both factors negative*
$(x-3) > 0$	$(x+3/2) < 0$
$x > 3$	$x < -3/2$

Therefore the values of x that satisfy this inequality will be only those values lying in the shaded range on the number scale below.



EXAMPLE II:

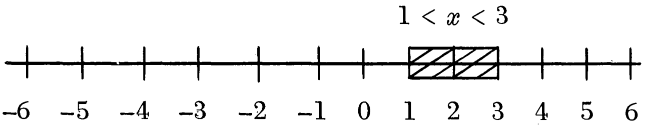
Solve the following inequality:

$$(2x-6)(3x-3) < 0 \ .$$

Dividing both sides of the inequality by 6, we obtain $(x-3)(x-1) < 0$. Since $(x-1) > (x-3)$, using Theorem II, we obtain the following solution.

Largest factor positive	*Smallest factor negative*
$(x-1) > 0$	$(x-3) < 0$
$x > 1$	$x < 3$

Therefore the values of x that satisfy this inequality will be only those values lying in the shaded range on the number scale below.



EXAMPLE III:

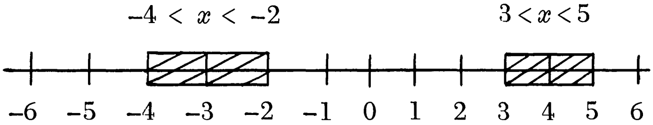
Solve the following inequality:

$$\frac{x^2-x-6}{x^2-x-20} < 0 \ .$$

This problem presents a special application of Theorem II. Obviously $(x^2-x-6) > (x^2-x-20)$. Using Theorem II we obtain the following solution.

Largest factor positive		*Smallest factor negative*	
$(x^2-x-6) > 0$		$(x^2-x-20) < 0$	
$(x-3)(x+2) > 0$		$(x-5)(x+4) < 0$	
Since $(x+2) > (x-3)$,		Since $(x+4) > (x-5)$,	
we obtain using Theorem I.		we obtain using Theorem II.	
Both factors +	*Both factors –*	*Larger factor +*	*Smaller factor –*
$(x-3) > 0$	$(x+2) < 0$	$(x+4) > 0$	$(x-5) < 0$
$x > 3$	$x < -2$	$x > -4$	$x < 5$

The values of x that satisfy the original inequality are only those values that satisfy both of the secondary inequalities obtained. These values of x are those values lying in the shaded regions on the number scale below.



EXAMPLE IV:

Solve the following inequality:

$$\frac{2x+3}{x^2} > \frac{2}{x-2}, \quad x \neq 0 \text{ and } x \neq 2 .$$

To solve this inequality first clear fractions recalling the following facts. $x^2 > 0$, for all $x \neq 0$ and $(x-2) > 0$ only if $x > 2$. Thus under these assumptions, we obtain,

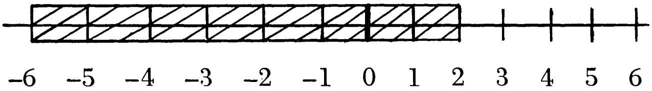
$$\begin{aligned}(2x+3)(x-2) &> 2x^2\\ (2x^2-x-6) &> 2x^2\\ (-x-6) &> 0\\ (x+6) &< 0\\ x &< -6\end{aligned}$$

This leads to a contradiction since we assumed to begin with that $x > 2$. So now assume that $(x-2) < 0$ or $x < 2$. After clearing fractions and changing the sense of the inequality, we obtain

$$\begin{aligned}(2x+3)(x-2) &< 2x^2\\ (2x^2-x-6) &< 2x^2\\ (-x-6) &< 0\\ (x+6) &> 0\\ x &> -6\end{aligned}$$

The values of x that satisfy this inequality will be only those values lying in the shaded area on the number scale below.

$$-6 < x < 2 \; ; \quad x \neq 0$$



This solution can better be written $-6 < x < 0$ and $0 < x < 2$.

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MISCELLANEOUS NOTES

Edited by

Roy Dubish

Articles intended for this department should be sent to *Roy Dubisch, Department of Mathematics, University of Washington, Seattle, Washington.*

A GENERALIZATION OF THE CONGRUENCE $r^x \equiv x$ (mod p)

Roger Osborn

For those who are unfamiliar with the theory of congruences in number theory, attention is directed to reference 1 which contains some explanatory remarks. In addition to the ideas explained briefly therein, three new ideas are used here. They are as follows. We say that the integer a belongs to the exponent e modulo p if $a^e \equiv 1 \pmod p$ and if there exists no smaller exponent i for which $a^i \equiv 1 \pmod p$. The statement that "a belongs to e modulo p" is made symbolically by $a \to e \pmod p$. The second new idea used herein is that of divisibility. We say that the integer a is divisible by the integer b if the quotient of a divided by b is an integer. The statement that "a is divisible by b" is symbolized by $b \mid a$ and is sometimes read "b divides a" — meaning an integral number of times. Also, we use the concept of number class. A number class is defined to be a set of integers all of which are congruent to each other modulo p.

The purpose of this paper is to obtain certain generalizations of the results set forth in reference 1. The generalizations here presented may be stated in the form of the following theorem.

Theorem. For each integer a satisfying $a < p$, $(a, d) = 1$, $a \to d \pmod p$, there exists an integer b for which $b^a \equiv a \pmod p$.

Proof. Since $a \to d \pmod p$,

$$a^d \equiv 1 \pmod p ,$$

$$a^{dx} \equiv 1 \pmod p ,$$

and

$$a^{dx+1} \equiv a \pmod p .$$

Require that $dx + 1 = ma$. This requirement is equivalent to requiring that the Diophantine equation

(1) $$am - dx = 1$$

possess solutions. Since $(a, d) = 1$ by hypothesis, and since this is a condition under which the Diophantine equation possesses solutions, solutions do exist. These are given by

(2) $$\begin{aligned} m &= m_0 + kd , \\ x &= x_0 + ka . \end{aligned}$$

The integer k may be chosen large enough to cause both m and x to be positive. Let m and x represent two such positive values. Then

$$a^{dx+1} \equiv a^{ma} \equiv a \pmod{p} ,$$

and

$$(a^m)^a \equiv a \pmod{p} . \tag{3}$$

Let $a^m \equiv b \pmod{p}$, which then completes the proof that, under the stated conditions, there exists at least one integer b for which

$$b^a \equiv a \pmod{p} . \tag{4}$$

It should be remarked here that if $d = p-1$, the congruence (4) reduces to the congruence treated in reference 1.

A few comments about the integers which satisfy (4) may now be made.

I. Any integer which is congruent to b modulo p also satisfies the congruence. That is, b is a representative member of a number class all members of which satisfy the congruence.

II. If a is an even integer, then all members of the number class of which $p-b$ is a member also satisfy the congruence. This may be seen as follows:

$$b^a \equiv a \pmod{p} ,$$

and since a is even,

$$(-1)^a b^a \equiv (-b)^a \equiv (p-b)^a \equiv a \pmod{p} .$$

III. Examination of equation (1) reveals that any common divisor of m and d would have to divide the right member of the equation. Hence, $(m, d) = 1$. Since $(m, d) = 1$, or, stated in words, since m is relatively prime to d, we find that b, which is congruent to a^m modulo p, must also belong to d. To see this we could assume that $b^i \equiv 1 \pmod{p}$ for some positive integer $i < d$. Then

$$b^i \equiv (a^m)^i \equiv a^{mi} \equiv 1 \pmod{p} .$$

That is, since $a \to d$, $mi = kd$ for some value of k. Since $(m, d) = 1$, then $d \mid i$, contrary to the assumption that $i < d$. Hence, no such i exists.

IV. Assume from above that $b^a \equiv a \pmod{p}$, remembering that $a \to d$ and $b \to d \pmod{p}$ and that $(a, d) = 1$. Also assume that there exists another integer c for which $c^a \equiv a \pmod{p}$, and assume, too, that $c \to d \pmod{p}$. If $c \to d \pmod{p}$, then c is in the period of a, that is, $c \equiv a^r \pmod{p}$ for some value of r. Now by definition, $b \equiv a^m \pmod{p}$, and therefore, $(a^m)^a \equiv a \pmod{p}$ and $(a^r)^a \equiv a \pmod{p}$. Assume $m < r$ for definiteness sake. (If $r < m$, the same results follow.) Then, by division, $a^{am-ar} \equiv 1 \pmod{p}$, or $a^{a(m-r)} \equiv 1 \pmod{p}$. Therefore, by a well known result of Fermat's theorem, $d \mid a(m-r)$. Since $(a, d) = 1$ from above, then $d \mid (m-r)$. Now m may be chosen initially $m < d$, and similarly for r. Therefore, $d \mid (m-r)$ is impossible unless $m-r = 0$, or $m = r$. Hence, $b \equiv a^m \equiv c \pmod p$, and b and c belong to the same number class. If each is expressed as a least positive residue, then $b = c$. Hence, no other integer which belongs to d and is distinct from b will satisfy a congruence $c^a \equiv a \pmod p$. Therefore, if there exists an integer b satisfying $b^a \equiv a \pmod p$ and an integer c satisfying $c^a \equiv a \pmod p$, then c does not belong to d modulo p. Let $c \to f$. Then $a^f \equiv c^{fa} \equiv 1 \pmod p$. Since $a \to d$, and $a^f \equiv 1 \pmod p$, then $d \mid f$. Therefore, if a value of c exists under the hypotheses, it belongs to an exponent which is a multiple of d. Looking back to remark II, if $b^a \equiv a \pmod p$, and if a is even, then $(p-b)^a \equiv a \pmod p$. We see, then, that $p-b$ belongs to an exponent which is a multiple of d.

Example. For $p = 19$, the following belonging relations exist. $1 \to 1$; $18 \to 2$; $7, 11 \to 3$; $8, 12 \to 6$; $4, 5, 6, 9, 16, 17 \to 9$; and $2, 3, 10, 13, 14, 15 \to 18$. It is apparent that 1; 7, 11; 4, 5, 16, 17; 13 satisfy the hypotheses of the theorem. These values of a give rise to the following congruences modulo 19: $1^1 \equiv 1$; $7^7 \equiv 7$; $7^{11} \equiv 11$; $6^4 \equiv 4$; $6^5 \equiv 5$; $5^{16} \equiv 16$; $9^{17} \equiv 17$; $10^{13} \equiv 13$. By use of remark II, two additional such congruences may be obtained. They are: $13^4 \equiv 4$ and $14^{16} \equiv 16$.

Note added in proof. That values of a do exist is obvious. Any of the $\phi(p-1)$ positive integers which are relatively prime to $p-1$, which are less than p, and which, since they are relatively prime to $p-1$, are relatively prime to the integers d (which are divisors of $p-1$) to which they belong can be values of a.

REFERENCE

1. Osborn, Roger, "Concerning the Congruence $r^x \equiv x \pmod p$, " THE TEXAS JOURNAL OF SCIENCE, Vol. XI, No. 3, 1959, pp. 270-274.

The University of Texas

PERMUTATION ORDERING AND IDENTIFICATION

Peter Shahdan

Introduction. The problem treated in this paper arose rather incidentally as a matter of curiosity in dealing with ordinary problems on permutations.

We all are familiar with the elementary fact of the total number of permutations which can be performed on n objects. Factorial n becomes amazingly large; and even when some of the objects are alike, the total number of permutations is still amazing! No wonder we use the exclamation symbol or 'wonder' mark!

It is natural to ask, as many of us have asked perhaps, is there some way of identifying any particular one of these permutations? Or, if any one of them is called for, can it be produced or derived?

The answer to both of these questions, as shown in what follows, is 'yes.'

We would like to examine the following two problems :

1. A general method of ordering and identifying the permutations in a set of permutations of n objects with or without like elements.

2. Also, the inverse process of obtaining any desired permutation of the set.

LEMMA I. If there are n objects, all different, the total number of $n!$ permutations may be divided into n groups, each group containing $\frac{1}{n}(n!)$ permutations.

Proof: If we place one of the n objects in the first position, then the remaining $(n-1)$ objects which follow it may be permuted in $(n-1)!$ ways. Since the first position may be filled in n ways, there will be n groups of $(n-1)!$ permutations each, or $n(n-1)! = n!$ permutations in all. But

$$(n-1)! = \frac{n!}{n} = \frac{1}{n}(n!) \ . \quad \text{Q.E.D.}$$

LEMMA II. If among n objects k are alike and $(n-k)$ different, then the total number of $\frac{n!}{k!}$ permutations may be divided into $(n-k+1)$ groups. In the group having one of the like elements in the first position, there will be $\frac{k}{n}(\frac{n!}{k!})$ permutations; in each of the other $(n-k)$ groups there will be $\frac{1}{n}\frac{(n!)}{k!}$ permutations.

Proof: For every one of the $(n-k)$ different objects which may be placed in the first position, there will be

$$\frac{(n-1)!}{k!} = \frac{1}{n}(\frac{n!}{k!})$$

permutations of the remaining objects. Hence there will be $(n-k)\frac{(n-1)!}{k!}$ of these permutations. Subtracting these from the total $n!/k!$ permutations in the set, we have

$$\frac{n!}{k!}-\frac{(n-k)(n-1)!}{k!}=\frac{n(n-1)!-(n-k)(n-1)!}{k!}=\frac{(n-1)!}{k!}[n-(n-k)]$$

$$=\frac{k(n-1)!}{k!}=\frac{k}{n}\frac{(n!)}{(k!)}. \quad \text{Q. E. D.}$$

THEOREM. If among n objects k are alike of one kind; m alike of another kind, etc., and $[n-(k+m+\cdots)]$ different, then the total number of permutations, $n!/k!m!\cdots$, may be divided into $[n-(k-1)-(m-1)-\cdots]$ groups.

The group which has one of the k like elements in the first position will contain $\frac{k}{n}\frac{(n!)}{(k!\,m!\cdots)}$ permutations.

The group which has one of the m like elements in the first position will contain $\frac{m}{n}\frac{(n!)}{(k!\,m!\cdots)}$ permutations. Etc.

And for each of the remaining $[n-(k+m+\cdots)]$ groups, there will be $\frac{1}{n}\frac{(n!)}{(k!\,m!\cdots)}$ permutations.

PROOF : From Lemma II, if one of the k alike objects is in the first position, then that group will contain $\frac{k}{n}\frac{(n!)}{(k!\,m!\cdots)}$ permutations. If one of the m alike objects is in the first position, then that group will contain $\frac{m}{n}\frac{(n!)}{(n!\,m!\cdots)}$ permutations. Etc. Each of the other groups will contain $\frac{1}{n}\frac{(n!)}{(k!\,m!\cdots)}$ permutations. Adding these together we obtain

$$\frac{k(n!)}{n(k!\,m!\cdots)}+\frac{m(n!)}{n(k!\,m!\cdots)}+\cdots+\left[\frac{n-(k+m+\cdots)}{n}\right]\frac{(n!)}{(k!\,m!\cdots)}=$$

$$\frac{n!}{n(k!\,m!\cdots)}(k+m+\cdots+[n-(k+m+\cdots)])=\frac{n!}{(k!\,m!\cdots)},$$

which is the total number of permutations in the set. Q. E. D.

Permutation Identification

In general, n objects can be permuted in $n!/k!m!\cdots$ ways. It is necessary to order these various permutations before identification might be possible. An easy and convenient method is to label the n objects with numerical subscripts: the k alikes with the digit 1; the m alikes with the digit 2; and the rest with the succeeding digits, 3, 4, 5, $\cdots$, in order. Then let the first arrangement of the n objects be $1111\cdots222\cdots345\cdots[n-(k-1)-(m-1)-\cdots]$. Any subsequent arrangement of the objects will yield an n-digit number of greater magnitude than the first. Hence let us order the permutations *according to the magnitude of the n-digit numbers* formed by the subscript labels.

The general method of identifying any given permutation will be in identifying the group and the sub-groups where each of the digits lies, and determining the number of permutations in each of the various groups.

In order to explain the 'modus operendi,' the following example is worked out in detail.

Example: Let $n = 6$; $k = 3$; $m = 2$; and the 6 th object different. We would then have $(6!/3!\,2!) = 60$ permutations, as follows.

1. 111223
2. 111232
3. 111322
4. 112123
5. 112132
6. 112213
7. 112231
8. 112312
9. 112321
10. 113122
11. 113212
12. 113221
13. 121123
14. 121132
15. 121213
16. 121231
17. 121312
18. 121321
19. 122113
20. 122131
21. 122311
22. 123112
23. 123121
24. 123211
25. 131122
26. 131212
27. 131221
28. 132112
29. 132121
30. 132211
31. 211123
32. 211132
33. 211213
34. 211231
35. 211312
36. 211321
37. 212113
38. 212131
39. 212311
40. 213112
41. 213121
42. 213211
43. 221113
44. 221131
45. 221311
46. 223111
47. 231112
48. 231121
49. 231211
50. 232111
51. 311122
52. 311212
53. 311221
54. 312112
55. 312121
56. 312211
57. 321112
58. 321121
59. 321211
60. 322111

Suppose now that we are given one of these 60 permutations, say

312121, and we are required to specify its cardinal number in the set, namely 55.

The first digit on the left, the 3, tells us that all the permutations which started with the smaller digits 1 and 2 have already preceeded it. According to lemma II, the group which started with the digit 1 contains 3/6 ths of the 60 permutations; 2/6 ths of the 60 permutations started with the digit 2. That means 50 permutations at least preceeded the given one; 30 with a 1 in the first position, and 20 with a 2 in the first position. Hence our given permutation must be the 5 th permutation in the group of permutations of the remaining 5 digits.

With the 3 in the first position, we now determine the 5 th permutation of the remaining 5 digits, namely 11122. Since the second digit in the given permutation is a 1, no smaller digit preceeded it; hence no permutations. We move now to the 3 rd digit, a 2. We now have to determine the 5 th permutation among the remaining digits 1122. According to our lemma II, 2/4 ths of $(4!/2!\,2!) = (2/4)(6) = 3$ permutations begin with the digit 1, and preceed the 2. Hence at least $50+3$ permutations preceeded the given one. We move now to the 4 th digit, a 1. Nothing preceeded the 1 in that position; hence no permutations. We move to the 5 th digit, a 2. Since we only have the 5 th and 6 th digits left, and since these two can only be in the order 12 or 21, we see that our given permutation has the second arrangement. Adding the two to our previous 53 preceeding permutations, we obtain 55, the cardinal number required.

Schematically, all this can be shown thus :

$$\frac{3\ \ 1\ \ 2\ \ 1\ \ 2\ \ 1}{5\ \ 0\ \ 2\ \ 0\ \ 1\ \ 0}$$

$$\frac{5(6!)}{6(3!\,2!)}+\frac{0(5!)}{5(3!\,2!)}+\frac{2(4!)}{4(2!\,2!)}+\frac{0(3!)}{3(2!)}+\frac{1(2!)}{2}+\frac{0(1!)}{1}+1=50+0+3+0+1+0+1$$
$$=55.$$

Note : The final unit is to account for the first permutation in the set, since it has no inversions of its digits.

Note : The numbers seen under the digits of the given permutation represent all the lesser digits which occupied that digital position previous to the digit which is now there. These numbers may be easily determined by counting all the lesser digits to the right of any of the digits in the given permutation. For instance, to the right of the first digit 3, there are the five lesser digits 1, 2, 1, 2, 1. Hence 5/6 ths of all the permutations preceeded the given one, at least. Similarly for the succeeding digits.

Conversely, given the cardinal number, P, of a permutation, find the permutation.

From the theorem, $\frac{k}{n}$ of the permutations begin with the digit 1. Hence if the given cardinal number,

$$P \leqq \frac{k}{n}\frac{(n!)}{(k!\,m!\cdots)}\ ,$$

then the first digit of the required permutation will be a 1. And if

$$\frac{k}{n}\frac{(n!)}{(k!\,m!\cdots)} < P \leqq \frac{(k+m)}{n}\frac{(n!)}{(k!\,m!\cdots)}\,,$$

then the first digit will be a 2. Etc.

Having determined the first digit, we proceed to determine the 2nd digit in a similar manner using the remaining $(n-1)$ digits as our base of calculations and with P diminished by the number of permutations accounted for by the first digit already established. Etc. for the 3rd, 4th, $\cdots$, nth digit.

Analytic Partitioning: A schematic process which we might call analytic partitioning, for obtaining the Pth permutation of a set of n objects with or without like elements.

For clarity's sake, let us work out the specific case already used where $n = 6$; $k = 3$; $m = 2$; and the 6th different; and $P = 55$.

Digits	*Cardinal numbers of the*				*Required*	
to be	*Permutations beginning with*			*P*	*Digits*	
Permuted	*Digit 1*	*Digit 2*	*Digit 3*			
111223	1-30	31-50	51-60	55	3	Row no. 1
11122	1-6	7-10		5	1	Row no. 2
1122	1-3	4-6		5	2	Row no. 3
112	1-2	3		2	1	Row no. 4
12	1	2		2	2	Row no. 5
1	1			1	1	Row no. 6
Col. 1	Col. 2	Col. 3	Col. 4	Col. 5	Col. 6	

Explanation: In row 1, the first item lists all the 6 elements of the problem. The second item lists the 30 permutations of the set which have a 1 as the first digit. The third item lists the next 20 permutations which have a 2 as the first digit. The fourth item lists the next 10 permutations which have a 3 as the first digit. The fifth item gives the cardinal number for the given set.

The last item in row 1 is a 3, determined by noting that 55 falls in the column of digit 3 where permutations 51 through 60 lie.

In row 2, the first item lists the remaining 5 digits of the original six. These 5 digits can be permuted in 10 ways; the first 6 of which are listed in the column for digit 1, and the remaining four permutations, 7 through 10, are listed in the column for digit 2. In the P space we put a 5 because that represents the remainder of the 55 original permutations, 50 of which are already accounted for by the first digit 3. Now since 5 falls in the column for digit 1, we place this digit 1 in the last column, making it the second required digit of the permutation we are seeking.

In row 3, we now have four of our original six digits left. As in rows 1 and 2, we fill in the numbers of the permutations of these remaining digits. The number in the P space remains a 5 because the previous digit, a 1, had no group or groups of permutations preceeding it. Hence nothing is subtracted from the previous P.

And similarly we proceed with the remaining rows.

Conclusion. It is not easy to say just how this solution may be utilized. Perhaps it can be applied in extended experiments where the order of the various steps or processes is of paramount importance, and it becomes necessary to repeat some.

N. C. State College
Raleigh, N. C.

A NUMERICAL CONGRUENCE

Charles W. Trigg

Norte de Problemas by J. Rey Pastor and J. Gallego-Diaz is an interesting 359-page collection of problems and solutions recently issued by Editorial DOSSAT, S. A., Madrid, Spain. The problem "Demonstrar que $63! \equiv 61!$ (mod 71). Generalizar." and its solution appear on pages 52 and 53. Wilson's theorem is used to arrive at a solution and at the generalization: "Si $(h-1)! \equiv -1 \pmod{p}$, siendo h par y p primo, se verifica: $(p-h)! \equiv -1 \pmod{p}$."

A more direct approach leads to a quicker solution and to a stronger but less erudite generalization. Note that

$$63! - 61! = (63 \cdot 62 - 1)(61!) = 5 \cdot 11 \cdot 71(61!) \equiv 0 \pmod{71}.$$

This suggests that if

$$h(h-1)(h-2)\cdots(h-k+1) \equiv 1 \pmod{x}$$

then

$$h! \equiv k! \pmod{x}$$

with no restrictions of primacy or parity upon h, k, or x.

Thus congruencies can be generated at will, e. g.,

$$9 \cdot 8 - 1 = 71, \quad \text{so} \quad 9! \equiv 7! \pmod{71};$$

$$19 \cdot 18 - 1 = 11 \cdot 31, \quad \text{so} \quad 19! \equiv 17! \pmod{31};$$

$$37 \cdot 36 - 1 = 11^3, \quad \text{so} \quad 37! \equiv 35! \pmod{11^3};$$

$$43 \cdot 42 - 1 = 5 \cdot 19^2, \quad \text{so} \quad 43! \equiv 41! \pmod{19^2};$$

$$11 \cdot 10 \cdot 9 - 1 = 23 \cdot 43, \quad \text{so} \quad 11! \equiv 8! \pmod{23};$$

$$17 \cdot 16 \cdot 15 \cdot 14 - 1 = 57119 \quad \text{so} \quad 17! \equiv 13! \pmod{57119}.$$

Los Angeles City College

CURRENT PAPERS AND BOOKS

Edited by H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to *H. V. Craig, Department of Applied Mathematics, University of Texas, Austin, 12, Texas.*

BOOK REVIEWS

Solution of Equations and Systems of Equations. By A. M. Ostrowski. Academic Press, New York, 1960, *ix* + 202 pp., $6.80.

Gathered into eighteen lectures and twelve supplementary appendices of Professor Ostrowski's new book are interesting results which have been published in diverse places and important results which have not been published previously. The main topics discussed are the method of false position, the method of iteration, and Newton's method. In the study of these major topics, the author develops and utilizes interpolation, inverse interpolation, Horner units, efficiency indices, points of attraction and repulsion, rates of convergence, existence theorems, Dandelin and Fourier bounds, fractional transformations, linear difference equations, norms of vectors and matrices, convergence and divergence of products of matrices, convergence and divergence of products of matrices, and various types of error analyses, including asymptotic behavior of errors. The extension of the theorem of Eneström and Kakeya, the establishment of an explicit formula for the nth derivative of the inverse function, the major theorems relating eigenvalues of certain matrices to points of attraction and repulsion, and the fundamental existence theorems for Newton's method are previously unpublished results which were of great interest to the reviewer. The absolute minimization of the number of necessary interpolation formulas, the very neat, repeated application of inverse interpolation to the calculation of errors, the appealing heuristic development of Steffensen's iteration formula to speed up convergence, and the various experienced remarks and hints on actual computing are further assets.

Without describing trivial typographical errors, a few dubious notational devices, and some moot rhetorical techniques, the following notes may still be of value in the reading of this first edition: p. 1, l. 7, note that J_x may be closed, open, or open at one end and closed at the other; p. 7, l. 4↑, "function" should read "polynomial"; p. 13, l. 3↑, "inside" should read "not outside"; p. 29, l. 6, "$x_c(x) = 1 - cF(x)$" should read "$x_c'(x) = 1 - cF'(x)$"; p. 40, equation (6.2), add the condition "$f'(a) \neq 0$"; p. 52, equation (8.10), "$(\nu = 0, 1, \cdots, p)$" should read "$(\nu = 0, 1, \cdots, p-1)$";

p. 67, l. 6, "will be at least quadratic" should read "will be, in general, at least quadratic"; p. 67, equation (11.2), add the condition "$\alpha\delta - \beta\gamma \neq 0$"; p. 73, l. 9, "$k_\mu = 0$" should read "$k_{\mu+n} = 0$"; p. 74, equation (12.8), add the condition "$\psi(x) \neq 0$"; p. 75, l. 4↑, note that this $K(x)$ is different from that of (12.5); p. 109, l. 8, "$= \lambda_A + \epsilon$" should read "$\leqq \lambda_A + \epsilon$"; p. 110, l. 10, "If in particular $\lambda_A < 1$, then···" should read "If in particular $\lambda_A < 1$ and $\epsilon < 1 - \lambda_A$, then···"; p. 118, l. 5↑, note that the definition of *point of repulsion* given here is not equivalent to that given on p. 26.

It is natural that the individual with some mathematical maturity will glean more from this book than the neophyte, for standard methods from matrix theory, complex variables, and, on one occasion, from very basic projective geometry are used to make certain discussions and proofs more succinct. Several casual paragraphs, each written in the pleasant tone of a friendly chat, are directed to readers who are experienced with computation. Regardless, Professor Ostrowski's new book will provide something of value to all those interested in the numerical solution of equations.

D. Greenspan

Differential Equations. By Tomlinson Fort. Holt, Rinehart and Winston, Inc., New York, 1960, *viii* + 184. $4.75.

This book represents an introductory course on differential equations at either the Freshman or the Sophomore level. We believe that, even in a fairly intense course, the book does not provide material for more than one semester.

In about 180 pages, the author covers from the basic definitions of a differential equation to linear partials. In between, such subjects as Laplace Transforms, matrices and boundary conditions are touched very briefly indeed. For example, about 8 pages are devoted to Laplace Transforms; vectors and matrices receive about 5 pages. The exposition is extremely clear, relying heavily on well chosen examples. The basic theorems of the theory of differential equations are proven and the nature of these proofs is commensurate with the general character of the book. Each chapter is followed by well selected exercises.

It seems that the main merit of this book consists of presenting introductory remarks and/or condensations of a vast field of analysis for students interested in mathematics merely as an auxiliary tool rather than a major subject. It should prove quite useful for students of engineering who do not wish to place heavy emphasis on the mathematical aspect of their education. The book is quite recommendable from the preceeding standpoint but is hardly more than an "aide-mémoire" for those who prefer to pursue the study of mathematics with more professional objectives.

Bernard G. Grunebaum

Introductory Analysis. By V. O. McBrien. Appleton-Century-Crofts, Inc., New York, 1961, $x + 188$ pp., $4.50.

This book was written to provide a one semester beginning course in mathematical analysis for students majoring in fields other than mathematics and the physical sciences. It attempts to provide motivation and insight into mathematics applied to such widely varying fields as economics, business, biology, and the behavorial sciences by using illustrations from these fields in the expository portions of the text as well as in problem sets.

The treatment on analysis is thoroughly modern. Starting with a brief introduction to sets, it develops the number continuum, then the Euclidean plane as a Cartesian product of linear sets. Common graphs in the plane are treated as sub-sets of E^2. Functions are defined as rules whereby a relation is established between two sets. Graphs of functions are identified with mappings. A brief mention of the topology of the real line leads into limits and thus into the concepts of calculus. Derivatives and Riemann integrals of algebraic and transcendental functions of one variable are covered and followed by n-space geometry and partial derivatives.

The book is a very interesting attempt to bring a very modern treatment of mathematics into a course serving the needs of those fields which have begun to apply mathematics to their problems only recently. This reviewer found the usual minor points to criticize. In the discussion of rational numbers, $a/0$ and $0/0$ were excluded from the set of rationals but no reason was given. Hence a good opportunity to throw light onto the number concept was missed.

Although the author uses many more problems from business, economics, and behavioral science than one finds in the traditional introductory analysis text, these fields must either be weak in problems which clarify the theory or the textbook writers have not found them. It is surprising to see that such a basic concept as rates of change is still illustrated in terms of applications from physical science.

The author and publishers are to be complimented on a pioneering effort in this area that is well worth a trial by those of us who are teaching students majoring in business, economics, and the behavioral sciences.

Robert E. Horton

BOOKS RECEIVED FOR REVIEW

Nuclear Reactor Theory. Proceedings of Symposia in Applied Mathematics, Vol. XI. Edited by G. Birkhoff and E. P. Wigner. American Mathematical Society, Providence, 1961, $v + 339$ pp.

A Modern Introduction to Logic. By Susan L. Stebbing. Harper and Brothers, New York, 1961, $xviii + 525$ pp. $2.75.

A Modern View of Geometry. By L. M. Blumenthal. W. H. Freeman and Co., San Francisco, 1961, *xii* + 191 pp. $2.25.

Sets, Logic and Axiomatic Theories. By R. R. Stoll. W. H. Freeman and Co., San Francisco, 1961, *x* + 206 pp. $2.25.

Measure, Lebesgue Integrals, and Hilbert Space. By A. N. Kolmogorov and S. V. Fomin. Academic Press, New York, 1961, *xii* + 147 pp. $4.00.

Elements of Linear Spaces. By A. R. Amir-Moez and A. L. Fass. Edwards Brothers, Inc., Ann Arbor, 1961, *vii* + 149 pp.

Partial Differential Equations and Continuum Mechanics. Edited by R. E. Langer. University of Wisconsin Press, Madison, 1961. $5.00.

Calculus. By Ivan Niven. Van Nostrand, New York, 1961, *viii* + 172 pp.

Tables of All Primitive Roots of Odd Primes Less Than 1000. By Roger Osborn. University of Texas Press, Austin, 1961, 70 pp. $3.00.

Modern Computing Methods. Edited by E. T. Goodwin. Philosophical Library, New York, 1961, *vi* + 170 pp. $6.00.

Classical Mathematics. By J. E. Hoffman. Philosophical Library, New York, 1959, 159 pp. $4.75.

Evaluation In Mathematics. Twenty-sixth Yearbook. National Council of Teachers of Mathematics, Washington, 1961, 215 pp. $3.00.

Secret Codes, Remainder Arithmetic and Matrices. By L. C. Peck. National Council of Teachers of Mathematics, Washington, 1961, *vi* + 54. $1.00.

Vectors in Three Dimensional Geometry. By A. M. Glickman. National council of Teachers of Mathematics, Washington, 1961, *vii* + 47 pp. $1.20.

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Ave., Los Angeles 29, California.*

PROPOSALS

453. *Proposed by Joseph W. Andrushkiw, Seton Hall University.*

a) If a root of the polynomial $f(x)$, whose roots are real, is of multiplicity three or greater, show that $F(x) = f(x) + c$, $c \neq 0$, cannot have all roots real.

b) The polynomials $f(x)$ and $g(x)$ have all real roots and $f(x) = g(x) + c$, $c > 0$. Prove that $h(x) = g(x) + k$, $0 < k < c$, has all roots real and distinct.

454. *Proposed by C. N. Mills, Sioux Falls College, South Dakota.*

Given the sides a and b and the included angle $c = 2\theta$ of a triangle. Prove that the length of the bisector of angle c is equal to $(2ab \cos \theta)/(a+b)$.

455. *Proposed by Leonard Carlitz, Duke University.*

Let $n > 1$. Show that:

a) $$x(x+1)\cdots(x+n-1) \equiv x^n - x \pmod{n}$$

if and only if n is prime;

b) $$\prod_{\substack{a=1 \\ (a,n)=1}}^{n} (x+a) \equiv x^{\phi(n)} - 1 \pmod{n}$$

if and only if $n = p$ or $2p$, where p is a prime.

456. *Proposed by M. S. Klamkin, AVCO, Wilmington, Massachusetts.*

Determine two parameter solutions of the following "almost" Fermat Diophantine equations:

1. $$x^{n-1} + y^{n-1} = z^n$$
2. $$x^{n+1} + y^{n+1} = z^n$$
3. $$x^{n+1} + y^{n-1} = z^n .$$

457. *Proposed by C. W. Trigg, Los Angeles City College.*

In a certain integer, the units' digit 6 is preceded by $(k-1)$ 5's, which in turn are preceded by k 1's. Find the square root of the integer.

458. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

A student used DeMoivre's theorem incorrectly as

$$(\sin \alpha + i \cos \alpha)^n = \sin n\alpha + i \cos n\alpha \ .$$

For what values of α does the equation hold for every integer n?

459. *Proposed by H. M. Gandhi, Lingraj College, Belgaum, India.*

Sum the series

$$\sum_{n=1}^{\infty} \left[\frac{n}{x^n} + \frac{n+1}{x^{n+1}} + \frac{n+2}{x^{n+2}} + \cdots\right] .$$

SOLUTIONS

Late Solutions

418, 419, 421, 422, 424, 425, 426, 427, 428, 429, 430, 431. *Josef Andersson, Vaxholm, Sweden;*
428, 429, 430, 431. *C. F. Pinzka, University of Cincinnati-*

A Triangular Inequality

423. [September 1960] *Proposed by David L. Silverman, University of Maryland.*

Prove that for real numbers a, b, and c,

$$|a-b| + |a+b-2c+|a-b|| < a+b$$

if and only if

$$|c-b| + |c+b-2a+|c-b|| < c+b \ .$$

Solution by I. Dale Ruggles, San Jose State College.

We need only do the only if case, for the proof of sufficiency follows by interchanging a and c in the first proof.

Given:

$$|a-b| + |a+b-2c+|a-b|| < a+b$$

it follows that

1) $a+b-2c+|a-b| < a+b-|a-b|$ or, $|a-b| < c$.

2) $-a-b+2c-|a-b| < a+b-|a-b|$ or, $|c-b| < a$.

From these, we get

$$|b-c| < a \quad \text{and} \quad a < b+c \ .$$

It then follows that

$$c+b-2a+|c-b| < c+b-|c-b| \quad \text{and} \quad 2a-|c-b|-b-c < b+c-|c-b| \ .$$

Therefore,

$$|c+b-2a+|c-b|| < b+c-|c-b|$$

and so $$|c-b| + |c+b-2a+|c-b|| < b+c\ .$$

Also solved by Josef Andersson, Vaxholm, Sweden; C. F. Pinzka, University of Cincinnati; and the proposer. One incorrect solution was received.

Cevian Lines

432. [January 1961] *Proposed by Lee Tih-Ming, Taipei, Taiwan.*

A point O interior to triangle ABC is joined to the vertices. From O perpendiculars OX, OY, OZ are dropped to the sides BC, CA, AB, respectively. AO and YZ intersect in D, BO and ZX in E, and CO and XY in F. Show that

$$\frac{AZ}{ZB}\cdot\frac{BX}{XC}\cdot\frac{CY}{YA}=\frac{ZD}{DY}\cdot\frac{YF}{FX}\cdot\frac{XE}{EZ}\ .$$

I. *Solution by C. F. Pinzka, University of Cincinnati.* Although the result follows directly from Ceva's theorem, it may also be proved as follows: Since A, Z, O, Y lie on the same circle, it is easily seen that triangles AZD and YOD are similar; DOZ and DYA are also similar. It follows that

$$\frac{AZ}{ZD}=\frac{YO}{OD},\quad \frac{DO}{OZ}=\frac{DY}{YA},\quad\text{and}\quad \frac{AZ}{ZD}\cdot\frac{DY}{YA}=\frac{YO}{OZ}\ .$$

In a similar manner,

$$\frac{BX}{XE}\cdot\frac{EZ}{ZB}=\frac{ZO}{OX}\quad\text{and}\quad \frac{CY}{YF}\cdot\frac{FX}{XC}=\frac{OX}{YO}\ .$$

Multiplying corresponding members of the last three equations together gives the desired result.

II. *Solution by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

The point O is not necessarily within the triangle. Letting

$$\alpha=\measuredangle BAO \qquad \beta=\measuredangle CBO \qquad \gamma=\measuredangle ACO$$
$$\alpha'=\measuredangle OAC \qquad \beta'=\measuredangle OBA \qquad \gamma'=\measuredangle OCB$$

we write from the triangles such as AZD and ADY, the relations

$$\frac{ZD}{\sin\alpha}=\frac{AZ}{\sin D},\quad \frac{DY}{\sin\alpha'}=\frac{YA}{\sin D}\quad\text{and}\quad \frac{ZD}{DY}=\frac{AZ}{YA}\cdot\frac{\sin\alpha}{\sin\alpha'}$$

and two others. Multiplying the three ratios member to member we obtain

$$\frac{ZD}{DY}\cdot\frac{YF}{FX}\cdot\frac{XE}{EZ}=\frac{AZ}{ZB}\cdot\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\left(\frac{\sin\alpha}{\sin\alpha'}\cdot\frac{\sin\beta}{\sin\beta'}\cdot\frac{\sin\gamma}{\sin\gamma'}\right).$$

But the expression in the parenthesis is 1, since AO, BO, CO are concurrent. Hence the equality is true for all points in the plane of ABC.

Also solved by Brother Alfred, St. Mary's College, California; Josef Andersson, Vaxholm, Sweden; C. W. Trigg, Los Angeles City College; Dale Woods, Oklahoma State University; and the proposer.

Downward Trajectory

433. [January 1961] *Proposed by C. W. Trigg, Los Angeles City College.*

A stone was thrown downward from a roof, at the level of the roof, at an angle of 30° with the horizontal. It passed the upper corner of a rectangular window at 45° with the horizontal and the opposite lower corner at an angle of 60° with the horizontal. If the path of the stone was in a plane parallel to the wall and the window was 6.0 ft. high, find

a) the width of the window;

b) the height of the roof above the window;

c) the speed with which the stone was thrown.

Consider the angles to be given to two-figure accuracy.

Solution by Peter Ploch, Wittenberg University, Ohio.

Let A (0,0) and B $(b,6)$ denote the locations of the stone as it passes the upper and the lower corners of the window, respectively. Routine solution to the projectile problem as presented in elementary calculus produces for the equation of the path: $y = x+(gx^2)/(V_A^2)$, where V_A denotes the velocity of the stone at A. Hence $y' = 1+(2gx)/(V_A^2)$. Using the given conditions at point B, we get equations

$$6 = b + \frac{gb^2}{V_A^2} \quad \text{and} \quad \sqrt{3} = 1 + \frac{2gb}{V_A^2} \,,$$

whose simultaneous solution is

$$b = 6(\sqrt{3}-1) \quad \text{and} \quad V_A = 2\sqrt{3g} \,.$$

Thus the equation of the path is $y=x+(x^2)/(12)$. At roof level, $y' = 1/\sqrt{3} = 1+x/6$, whence $x = 2\sqrt{3}-6$ and $y = -2$. Finally, since the horizontal component of velocity is constant, letting V_0 denote the velocity with which the stone was thrown from the roof, we must have $V_0 \cos 30° = V_A \cos 45°$ or $V_0 = 2\sqrt{3g}\cos 45° \sec 30° = 2\sqrt{2g} = 16$. Thus the roof is two feet above the window, which is $6(\sqrt{3}-1) = 4.4$ feet in width, and the initial velocity was $2\sqrt{2g} = 16$ ft./sec.

Also solved by Josef Andersson, Vaxholm, Sweden; Robert H. Clark, U. S. Naval Underwater Ordinance Station, Rhode Island; Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; J. W. Mellender, University of Wisconsin; C. F. Pinzka, University of Cincinnati; P. D. Thomas, U. S. Coast and Geodetic Survey, Washington, D. C.; Roger Wade, Brown University; and the proposer.

Illegal Cancellation

434. [January 1961] *Proposed by B. L. Schwartz, Technical Operations, Inc., Honolulu, Hawaii.*

The common fractions 19/95, 26/65, 16/64, and 49/98 can all be reduced to lower terms by "cancelling" the common digits in the numerator and denominator. These are the only proper fractions with two-digit denominators with this property. Characterize the proper fractions with denominators less than 1000 which yield to the same incorrect method.

Solution by C. W. Trigg, Los Angeles City College.

For the sake of clarity, illegal "cancelling" is defined as the striking out of identical digits in the numerator and denominator of a fraction, the remaining digits being interpreted as constituting the resultant fraction. The obvious exceptions occur when both unit's digits, or both unit's digits and both hundred's digits are zero, in which cases the cancellation of the zeros is legitimate.

For three-digit denominators, the numerator may contain two digits only one of which may be cancelled, or three digits of which one or two may be cancelled. We restrict the problem further by requiring that the cancellation reduce the fraction to *lowest* terms.

Case 1. The class of fractions having two-digit numerators may be represented by $(10a+b)/(100d+10e+f)$ in which $a \neq 0$, $d \neq 0$. Either a or b may be cancelled with any one of d, e, or f. Each consequent fractional equality constitutes a Diophantine equation in four unknowns. No solutions exist for $a = f$ or for $b = f$. The other sub-cases yield 37 solutions, some of which can be cancelled and reduced to *lowest* terms, in two ways. The original fractions are: 11/110, 21/210, 31/310, 41/410, 51/510, 61/610, 71/710, 81/810, 91/910, 12/120, 13/130, 14/140, 15/150, 16/160, 17/170, 18/180, 19/190, 22/121, 44/143, 55/154, 77/176, 88/187, 55/253, 77/275, 13/325, 44/341, 55/352, 77/374, 88/385, 55/451, 77/473, 77/572, 88/583, 19/950, 16/640, 77/671, and 88/781.

Case 2. In the fraction $(100a+10b+c)/(100d+10e+f)$, there are (3)(3) or 9 ways in which one digit of the numerator can be cancelled with one digit of the denominator. Each consequent fractional equality is a Diophantine equation in five unknowns. There are no solutions for $a=d$, $a=f$, $b=f$, $c=d$, $c=e$, or $c=f$. The other sub-cases yield 48 solutions, ten of which may be cancelled in two ways. The original fractions, in each of which $a+c = b = e = d+f$, are: 121/220, 143/242, *143/341, 242/341, 143/440, 341/440, 154/253, *253/352, *154/451, 253/451, 352/451, 253/550, 451/550, 176/275, 275/374, 176/473, 275/473, *374/473, *275/572, 473/572, *176/671, 275/671, 374/671, 473/671, 572/671, 473/770, 671/770, 187/286, 187/385, 286/385, 187/484, 385/484, 187/583, 286/583, *385/583, 484/583, 187/682, 385/682, 583/682, *187/781, 286/781, 385/781, 484/781, 583/781, 682/781, 187/880, 583/880, 781/880. In eight cases, those marked with an asterisk (*), the numerator and denominator are palindromes.

Case 3. In the fraction $(100a+10b+c)/(100d+10e+f)$ there are 18 ways in which two digits of the numerator may cancel with two digits of the denominator. There are no solutions for $a=d$, $b=e$; $a=d$, $b=f$; $a=d$, $c=f$; $a=e$, $c=f$; $a=f$, $b=d$; $a=f$, $b=e$; $a=f$, $c=d$; or $b=e$, $c=f$. The other sub-cases yield 28 solutions, nine of which may be cancelled in two ways. The following original fractions are grouped to show that sometimes the value of the fraction is unchanged when certain shifts in the orders of the digits are made: 125/217 = 412/721, 182/819 = 218/981, 163/326 = 316/632, 273/728 = 327/872, 244/427 = 424/742, 364/637 = 436/763, 448/784 = 484/847, 455/546 = 545/654, 127/762, 138/184, 139/973, 145/435, 148/185, 160/640, 166/664, 187/748, 190/950, 199/995, 260/650, 266/665.

Thus, including the three fractions in the proposal, there are 116 proper fractions with denominators less than 1000 that may be reduced to *lowest* terms by illegal ''cancellation.''

Eighty of these fractions were given by W. E. Buker in his solution to problem 1317, *School Science and Mathematics*, 34 (April 1934), p. 432-3. The original problem, wherein the denominator is less than 100, appeared as problem E 24, *American Mathematical Monthly,* 40 (August 1933), p. 425.

Also solved by Josef Andersson, Vaxholm, Sweden (Partially) and Brother Alfred, St. Mary's College, California (Partially).

Triangular Extrema

435. [January 1961] *Proposed by M. S. Klamkin, AVCO, Wilmington, Massachusetts.*

Determine the largest and the smallest equilateral triangles that can be inscribed in an ellipse.

Solution by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.

Let $A_1A_2A_3$ be an equilateral triangle inscribed in the ellipse

$$(1) \qquad (x^2/a^2)+(y^2/b^2)=1 \quad (E) \quad a>b$$

and let

$$(2) \qquad (x-u)^2+(y-v)^2-r^2=0 \quad (\Omega)$$

be the circle circumscribed to $A_1A_2A_3$. It cuts (E) at the fourth point $A_4(x_4, y_4)$.

Eliminating y between (1) and (2) we get an equation of fourth degree in x

$$c^4\cdot x^4-4a^2c^2u\cdot x^3+\cdots=0$$

of which the roots are x_1, x_2, x_3, x_4.

If we elimate x between (1) and (2), the corresponding equation will be

$$c^4\cdot y^4+4b^2c^2v\cdot y^3+\cdots=0$$

and the roots are y_1, y_2, y_3, y_4.

Since $A_1A_2A_3$ is an equilateral triangle, we have

$$x_1+x_2+x_3 = 3u$$

$$y_1+y_2+y_3 = 3v$$

and

$$\begin{aligned} x_4 &= \sum x_i - 3u = \frac{4a^2u}{c^2} - 3u = \frac{(a^2+3b^2)u}{c^2} \\ y_4 &= \sum y_i - 3v = -\frac{4b^2v}{c^2} - 3v = -\frac{(b^2+3a^2)v}{c^2} \;. \end{aligned} \tag{3}$$

The coordinates (3) satisfying (1) we obtain the relation

$$(u^2/\alpha^2)+(v^2/\beta^2) = 1 \tag{4}$$

where

$$\alpha = \frac{ac^2}{a^2+3b^2}\,, \quad \beta = \frac{bc^2}{b^2+3a^2}\,.$$

Hence the centers of the circles (Ω) lie on the ellipse (4) of which $\alpha > \beta$.

Now since the largest and the smallest triangles correspond to the greatest and the smallest values of the radius r of the circle (Ω), we write

$$\begin{aligned} r^2 &= (x_4-u)^2+(y_4-v)^2 \\ &= \frac{(a-\alpha)^2u^2}{\alpha^2} + \frac{(b+\beta)^2v^2}{\beta^2} \\ &= Au^2+(b+\beta)^2 = Bv^2+(a-\alpha)^2\,. \end{aligned}$$

$dr/du = 0$ gives

$$u = 0 \quad \text{and} \quad r_1 = b+\beta\,.$$

Similarly $dr/dv = 0$ gives

$$r_2 = a-\alpha\,,$$

and one may readily verify that $r_1 > r_2$.

Hence, the largest (smallest) equilateral triangles inscribed in the ellipse, are ones inscribed to the circles of center $u = 0$, $v = \pm\beta$ ($u = \pm\alpha$, $v = 0$) and radius $b+\beta$ ($a-\alpha$).

There are four solutions, two for the largest and two for the smallest triangles.

Constructions: The largest (smallest) triangles inscribed in an ellipse, have one of their vertices at the extremities of the minor (major) axis of the ellipse, the axis being the axis of symmetry of the triangle.

Also solved by Josef Andersson, Vaxholm, Sweden; J. W. Clawson, Collegeville, Pennsylvania (Two solutions); and J. W. Mellender, University *of Wisconsin.*

A Series Solution

436. [January 1961] *Proposed by Souren Babikian, Los Angeles City College.*
If

$$\tan(\phi+\theta) = \frac{3\tan\theta+\tan^3\theta}{1+3\tan^2\theta},$$

show that one value of ϕ is the series

$$\frac{1}{1\cdot 3}\sin 4\theta + \frac{1}{2\cdot 3^2}\sin 8\theta + \frac{1}{3\cdot 3^3}\sin 12\theta + \cdots .$$

I. *Solution by P. D. Thomas, U. S. Coast and Geodetic Survey, Washington, D. C.*

With the substitution $\tan(\phi+\theta) = (\tan\phi+\tan\theta)/(1-\tan\phi\tan\theta)$ in the left member of the given equation we find that

$$(1) \qquad \tan\phi = \frac{2\tan\theta(1-\tan^2\theta)}{1+6\tan^2\theta+\tan^4\theta} = \frac{\sin 4\theta}{\{1/(1/3)\}-\cos 4\theta}.$$

Now it is known that if $\tan\phi = \sin A/\{(1/n)-\cos A\}$, $n < 1$, then

$$(2) \qquad \phi = n\sin A + \frac{1}{2}n^2\sin 2A + \frac{1}{3}n^3\sin 3A + \cdots, \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2}.$$

(For instance see *Plane Trigonometry*, Loney, 1915 Edition, p. 129.) From (1) $n = 1/3$, $A = 4\theta$ and these values placed in (2) give the announced expansion.

II. *Solution by L. Carlitz, Duke University.*

It follows easily from

$$\tan(\phi+\theta) = \frac{3\tan\theta+\tan^3\theta}{1+3\tan^2\theta}$$

that

$$(*) \qquad \tan\phi = \frac{\sin 4\theta}{3-\cos 4\theta}.$$

Now if

$$w = -\log(1-\frac{1}{3}e^{4i\theta}) = \sum_{n=1}^{\infty}\frac{e^{4ni\theta}}{n\cdot 3^n},$$

then

$$v = I(w) = \sum_{n=1}^{\infty}\frac{1}{n\cdot 3^n}\sin 4n\theta .$$

But since

$$-i\tan v = \frac{e^{-w} - e^{-\overline{w}}}{e^{-w} + e^{-\overline{w}}} = \frac{(1-\frac{1}{3}e^{4i\theta}) - (1-\frac{1}{3}e^{-4i\theta})}{(1-\frac{1}{3}e^{4i\theta}) + (1-\frac{1}{3}e^{-4i\theta})} = \frac{\frac{2i}{3}\sin 4\theta}{2-\frac{2}{3}\cos 4\theta},$$

we get

$$(**) \qquad \tan v = \frac{\sin 4\theta}{3-\cos 4\theta}.$$

Comparing (**) with (*), the stated result follows at once.

Also solved by Josef Andersson, Vaxholm, Sweden; Marian T. Bird, San Jose State College; Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; and the proposer.

A Well Known Problem

437. [January 1961] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Prove or disprove the statement: The number of odd coefficients in the binomial expansion of $(a+b)^{[n]}$ is a power of 2, the exponent of 2 being the number of 1's appearing in the expression of n in the binary number system.

Editor's note: Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania, pointed out that a simpler version of this problem appeared as Problem 7, Part II, in the Putnam Competition of 1956. The given problem appeared as Problem E 1288 in the *American Mathematical Monthly* in November 1957 with solution and references given in the May, 1958 issue.

A Triangular Configuration

438. [January 1961] *Proposed by Leon Bankoff, Los Angeles, California.*

Let D be the apex of the equilateral triangle constructed externally on side BC of a triangle ABC, and let Q be the apex of the equilateral triangle constructed on AD, with Q and B on opposite sides of AD. Then let E be the apex of the equilateral triangle on the base CQ, with D and E on opposite sides of CQ. Finally, let F be the apex of the equilateral triangle on the base DE, with Q and F on opposite sides of DE. Show that triangle BAF is equilateral.

Solution by Josef Andersson, Vaxholm, Sweden. (Translated and paraphrased by the editor.)

Let a, b, c, $\cdots$ be the affixes of A, B, C, $\cdots$ in the Gaussian plane, and $\epsilon = e^{2\pi i}$, where ϵ satisfies $\epsilon^2+\epsilon-1 = 0$ and $\epsilon^3 = 1$. After a convenient orientation to the axes we have successively:

$$d = c-\epsilon(b-c) = \epsilon b - \epsilon^2 c\ ,$$

$$q = a-\epsilon(-\epsilon b-\epsilon^2 c-a) = -\epsilon^2 a+\epsilon^2 b+c$$

$$e = c-\epsilon(-\epsilon^2 a+c^2 b+c-c) = a-b+c$$

(where $ABCE$ is a parallelogram)

$$f = -\epsilon b-\epsilon^2 c-\epsilon(a-b+c+\epsilon b+\epsilon^2 c) = -\epsilon a-\epsilon^2 b = b-\epsilon(a-b)\ .$$

This proves the theorem.

Lemma. Given $KLMN$ is a parallelogram; KLU, KNV are equilateral such that the orientations $K \to L \to U \to K$ and $K \to N \to V \to K$ are opposite. Then MUV is equilateral and of the same orientation as KLU. In effect, if we rotate ULM about U through an angle of 60°, then UL becomes UK, LM becomes KV, and $UM = UV$, and angle $MUV = 60°$.

Conversely if we construct the equilateral triangles KLM and MUV with the same orientation, we have triangle KNV also equilateral with $KN = LM$ and the orientations of KLU and KNV are opposite. If we refer to the original figure we conclude 1) that $\overrightarrow{CE} = \overrightarrow{BA}$ [triangles CBD, ADQ] and then, 2) that BAF is equilateral [triangles BCD, EDF].

It might be preferable to establish the orientation of the various given triangles. Also the word "externally" may be replaced by "internally."

Also solved by J. W. Clawson, Collegeville, Pennsylvania; Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; P. D. Thomas, Coast and Geodetic Survey, Washington, D. C.; C. W. Trigg, Los Angeles City College; and the proposer.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 286. Find the nth derivative of $\cos^3 x$. [*Submitted by M. S. Klamkin.*]

Q 287. A series of books was published at seven-year intervals. When the seventh book was issued, the sum of the publication years was 13524. When was the first book published? [*Submitted by C. W. Trigg.*]

Q 273. [January 1961] *Correction by C. W. Trigg, after a nudge by James H. Hill, Jr.*

The quickie should have read, "The product of four consecutive odd integers is 3317184009," not 3313036881 as originally printed. Indeed, if the product had been given as 33*******9, the analysis of **A 273** would have yielded the same result, namely: (237)(239)(241)(243).

Q284. [May 1961] Comment by *Vladimir F. Ivanoff.*

In a plane, the locus of points whose projections on the sides of a triangle are collinear is not only a circle. The other part of the locus is the line infinity. Indeed, using trilinear coordinates, and taking the given triangle for reference, we set the condition for collinearity as follows:

$$\begin{vmatrix} 0 & y+x\cos C & z+x\cos B \\ x+y\cos C & 0 & z+y\cos A \\ x+z\cos B & y+z\cos A & 0 \end{vmatrix} = 0\,.$$

Using the identities:

$$\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A \cdot \cos B \cdot \cos C\,,$$
$$\cos A + \cos B \cdot \cos C = \sin B \cdot \sin C$$

for the angles of the triangle, we factorize our determinant as follows:

$$(xy\cdot\sin C + yz\cdot\sin A + zx\cdot\sin B)(x\cdot\sin A + y\cdot\sin B + z\cdot\sin C) = 0$$

where the first factor, equated to zero, represents the circumcircle of the triangle, and the second – the line infinity.

T44. [May 1961] *Comment by Hills Lee.*

The nomograph could be made more general by extending the axes R, r_1, and r_2 to include negative values of all three variables.

(Answers to the Quickies are on page 352.)

DEL LEMMA

Although life is a function of many unknowns
And its surfaces prone to deflection,
It is well to assume that a gradient exists
And attempt to deduce its direction.

Marlow Sholander

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ANSWERS *to Quickies on page 372.*

A 286.

$$D^n \cos^3 x = D^n\left[\frac{\cos 3x + 3\cos x}{4}\right] = \frac{3^n}{4}\cos(3x + n\pi/2) + (3/4)\cos(x + n\pi/2)\ .$$

A 287. The arithmetic mean of the publication years is 13524/7 or 1932. This middle term is separated from the first term by three common differences, so the first book was published in 1932 − 3(7) or 1911.

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